THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America met at George Washington University, Washington, D. C., morning and afternoon, December 6, 1919. Among those in attendance were the following members: H. L. Hodgkins, George Washington University; G. A. Bingley, C. C. Bramble, J. A. Bullard, G. R. Clements, G. H. Cresse, Alexander Dillingham, J. B. Eppes, P. E. Hemke and R. E. Root, U. S. Naval Academy; A. E. Landry and O. J. Ramler, Catholic University; Abraham Cohen and F. D. Murnaghan, Johns Hopkins University; Clara L. Bacon and Florence P. Lewis, Goucher College; Harry English, Washington Public Schools; H. M. Roeser, Bureau of Standards; W. E. Heal, Bureau of Plant Industry; J. J. Arnaud, A. A. Bennett and C. E. Norwood, Ordnance Department (Army); W. M. Hamilton, Nautical Almanac Office; O. S. Adams, H. G. Avers, Sarah Beall, W. D. Lambert and G. F. Winslow, Jr., Coast and Geodetic Survey.

The program contained the following papers:

(1) "Calculation of the date of Easter," Professor F. D. MURNAGHAN, Johns Hopkins University.

(2) "Precise leveling in the United States," Mr. H. G. AVERS, Coast and Geodetic Survey.

(3) "Proof that in a plane world infinite in extent, but finite in thickness, gravity would be a constant at any altitude," Mr. A. S. HAWKESWORTH, Naval Ordnance.

(4) "Geometrical proportion," Mr. W. E. HEAL, Bureau of Plant Industry.

(5) "Desirable changes in the mathematical courses in the high school," Mr. HARRY ENGLISH, Head of Mathematics Department, Washington High Schools.

(6) "Report of the Summer Meeting of the Association at Ann Arbor, Mich.," Dr. G. H. CRESSE, U. S. Naval Academy, and Professor FLORENCE P. LEWIS, Goucher College.

(7) "Internal constitution of the earth," Mr. W. D. LAMBERT, Coast and Geodetic Survey.

(8) "A simple instrument for the inversion and the mechanical calculation of trigonometric functions," Dr. T. DANZIG, Johns Hopkins University.

Interesting general discussions followed the reading of the various papers.

OSCAR S. ADAMS, *Secretary.*

THE THIRD ANNUAL MEETING OF THE MISSOURI SECTION.

The third annual meeting of the Missouri Section of the Mathematical Association of America, postponed from 1918 on account of the war, was held in St. Louis on the afternoons of Monday and Tuesday, December 29–30, 1919. The meeting would normally have been held in Columbia on the Saturday following Thanksgiving Day, in conjunction with the meeting of the Southwestern Section of the American Mathematical Society, but the time and place of meeting were changed so that these two sections might meet with the Chicago Section of the American Mathematical Society during convocation week of the American Association for the Advancement of Science. The meeting next year is scheduled to be held in Kansas City, and the meeting two years hence, which was to have been held in St. Louis, will take place in Columbia.

The Monday session, presided over by Professor William H. Roever, Chairman of the Section, was for the reading of papers and for the annual business meeting. The Tuesday session was a joint meeting with the Chicago and Southwestern Sections of the American Mathematical Society and Section A of the American Association for the Advancement of Science, and was presided over by Professor O. D. Kellogg, Chairman of Section A and Vice-Chairman of the Missouri Section. On Tuesday evening at the American Hotel Annex the four mathematical sections already mentioned held a joint dinner at which forty-eight persons were present.

At the Monday meeting there was an attendance of twenty-two, including the following members of the Mathematical Association of America: Charles Ammerman, McKinley High School, St. Louis; G. D. Birkhoff, Harvard University; J. A. Caparo, University of Notre Dame; Alfred Davis, Soldan High School, St. Louis; E. L. Dodd, University of Texas; Otto Dunkel, Washington University; H. J. Ettlinger, University of Texas; E. R. Hedrick, University of Missouri; Louise H. Huff, St. Louis; A. R. Nauer, St. Louis; P. R. Rider, Washington University; Percival Robertson, The Principia, St. Louis; W. H. Roever, Washington University; A. J. Schwartz, Cleveland High School, St. Louis; H. E. Slaught, University of Chicago; G. W. Smith, University of Kentucky; E. H. Taylor, Eastern Illinois State Normal School, Charleston, Ill.; W. H. Zeigel, State Normal School, Kirksville, Mo.

At the business meeting the report of the Committee appointed to consider the Preliminary Report on the Reorganization of the First Courses in Secondary School Mathematics by the National Committee on Mathematical Requirements was adopted.

It was voted that the Section urge the National Committee on Mathematica Requirements to take positive steps toward having speakers in behalf of mathematics address general rather than mathematical meetings.

The Section went on record as urging that a minimum of at least one year of

high school mathematics be required for graduation by all high schools in the state of Missouri.

Professor R. A. WELLS of Park College was elected Chairman and Mr. W. A. LUBY, of the Kansas City Polytechnic Institute, Vice-Chairman for the coming year. It was voted that the office of Secretary-Treasurer be considered permanent, and the present officer was reëlected.

It was voted that a committee be appointed by the new chairman to act on the reports of the National Committee on Mathematical Requirements. This committee has been appointed as follows: Professor E. R. Hedrick, University of Missouri, chairman; Professor W. H. Zeigel, State Normal School, Kirksville; Professor R. R. Fleet, William Jewell College; Miss Zoe Ferguson, Central High School and Junior College, St. Joseph; Mr. Alfred Davis, Soldan High School, St. Louis; Mr. Percival Robertson, The Principia, St. Louis.

At the joint meeting on Tuesday the following addresses were given:

I. "Recent advances in dynamics."[1] Address of the retiring Vice-President of Section A of the A. A. A. S., PROFESSOR G. D. BIRKHOFF.

II. "Some recent developments in the calculus of variations." Address of the retiring chairman of the Chicago Section of the American Mathematical Society, PROFESSOR G. A. BLISS, University of Chicago.

III. "A suggestion for the utilization of atmospheric molecular energy." MR. H. H. PLATT, Philadelphia.

At the regular session of the Missouri Section the following seven papers were read:

(1) Opening address as President of the Association, PROFESSOR H. E. SLAUGHT, University of Chicago.

(2) "The determination of logarithmic formulas," PROFESSOR E. R. HEDRICK, University of Missouri.

(3) "A Simple Treatment of Fourier's Series," PROFESSOR LOUIS INGOLD, University of Missouri, and MR. T. W. JACKSON, Jamestown College, N. D.

(4) "An Elementary Method of Quadrature," PROFESSOR OTTO DUNKEL, Washington University.

(5) "Plans of the National Committee on Mathematical Requirements," MR. CHARLES AMMERMAN, McKinley High School, St. Louis.

(6) "Preliminary report of the National Committee on Mathematical Requirements," PROFESSORS HEDRICK, ZEIGEL, and FLEET, MISS ZOE FERGUSON and MR. ALFRED DAVIS.

(7) "Geometric treatment of certain optical problems," (Illustrated by lantern views and models.) PROFESSOR WM. H. ROEVER, Washington University, chairman of the Section.

In the absence of the authors the paper by Professor Ingold and Mr. Jackson was read by Professor Hedrick. Abstracts of the papers, except (5) and (6), follow below, the numbers corresponding to the numbers in the list of titles above:

[1] Published in *Science*, January 16, 1920, pp. 51–55.

(1) Professor Slaught's address dealt with the program of the Association's plans, especially as related to the work of the Sections.

(2) Given a phenomenon subject to a law of the form $y = ax^n$, if we make the usual transformation $X = \log x$, $Y = \log y$, $A = \log a$, and then determine the line $Y = A + nX$ by the usual theory of least squares, there is an obvious exaggeration of the errors for small values of x. To correct this, Professor Hedrick suggested that the errors in Y be loaded by a factor given by the law of the mean, from which we find approximately

$$\frac{y - y_i}{Y - Y_i} = y_i,$$

where (x_i, y_i) are the values of x and y given by experiment. Then the expression which is to be made a minimum becomes

$$S = \sum [(A + nX_i) - Y_i]^2 y_i^2,$$

where (X_i, Y_i) are the values of X and Y corresponding to (x_i, y_i).

The remainder of the usual work can be written down very easily by comparison with the usual theory. In work done upon actual experimental data, this formula has proved to be convenient and to give results that are more satisfactory than those given by the usual theory.

Similar changes in the theory of least squares can be made to counteract the distortion caused by any other transformation, in a similar manner.

Finally, the connection was pointed out between this subject and the question of approach of a variable curve to a fixed curve. An extension of the principle mentioned above gives a satisfactory measure of the distance between two curves under analogous circumstances.

(3) The object of the paper of Professor Ingold and Mr. Jackson was to give a relatively simple proof of the convergence of the sine or cosine series, whose coefficients are determined in the Fourier manner, for a large class of functions including the usual examples; and also to prove that these series actually represent the functions from which they are obtained.

Their final result may be stated thus: Any function which may be expressed as a definite integral from 0 to x of a bounded integrable function may be represented by a sine or cosine series of the Fourier type for values of x between 0 and π.

For this class of functions the proof of convergence and representability is much simpler than the usual proofs.

(4) In the presentation of the summation formula in the integral calculus it is desirable to give an example of the evaluation of an area by finding the common limit of the sums of the areas of the inscribed and circumscribed rectangles. The usual examples of the texts involve the summation of series, which is quite difficult and unfamiliar to the student. Professor Dunkel gave a method for finding the area under curves $y = x^m$, m = a positive integer, which follows rather closely the lines of presentation of the summation formula.

The series to be actually summed is such that the sum is obvious and the whole process involves only elementary algebra.

The method can be extended to negative values of m, $m = -1$ being excluded, and also to fractional values. Compare *1920*, 116–117.

(7) The object of Professor Roever's paper was to illustrate a purely geometric method of treatment by its application to the solution of some problems in geometric optics. The problems treated in this way were certain problems on brilliant points which have already been treated analytically by the author in *Annals of Mathematics*, second series, vol. 3, April, 1902; *Transactions of the American Mathematical Society*, vol. 9, July, 1908; and AMERICAN MATHEMATICAL MONTHLY, vol. 20, December, 1913. An example of this geometric method is given in an article entitled: "Geometric Description of the Halo on the Dome of the St. Louis Cathedral," which will soon appear in a scientific number of the *Washington University Studies*.

PAUL R. RIDER, *Secretary-Treasurer*.

ENVELOPE ROSETTES.[1]

By WILLIAM F. RIGGE, Creighton University, Omaha, Neb.

One way of drawing a cardioid is to make a pen start in phase 90° from the center of a disk and move along a radius with simple harmonic motion, while the disk revolves with a uniform angular speed of the same period. If now, instead of a *simple* harmonic movement with the equation $\rho = 1 - \cos\theta$, the amplitude being unity, we give the pen a *double* harmonic motion, and write the equation

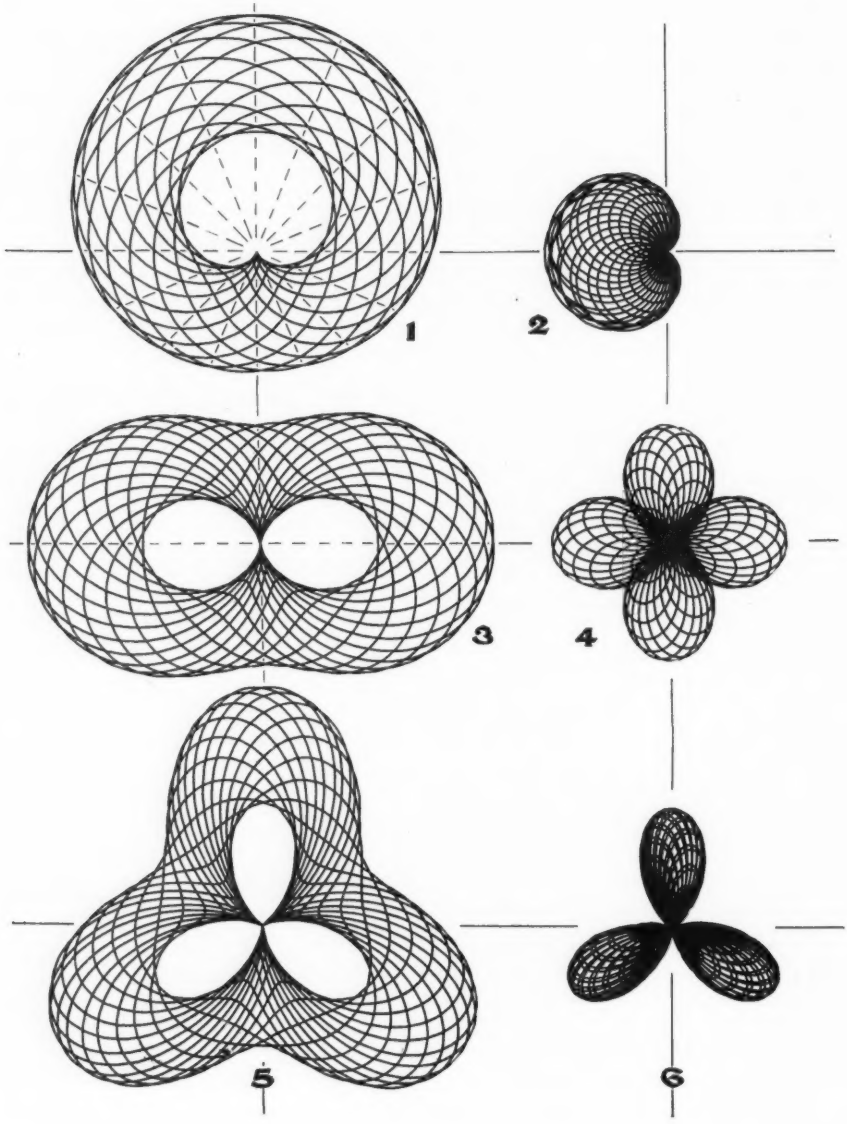
$$\rho = (1 - \cos\theta) + (1 - \cos m\theta),$$

in which m differs from unity by some small aliquot fraction; we shall then get a series of harmonic curves which have one variable parameter, and which must therefore have a common envelope. The problem before us is to find this envelope.

The Inner Envelope a Cardioid.—The method of procedure, according to the textbooks, is to differentiate the above equation by regarding the variable parameter m as the only variable in it, and then to eliminate the parameter between these two equations. This will give us

$$\sin m\theta = 0,$$

[1] Readers of this article are reminded of earlier articles by Professor Rigge in this MONTHLY ("Concerning a new method of tracing cardioids," *1919*, 21–32; "Cuspidal rosettes," *1919*, 332–340) in which the discussion, with special reference to possibilities of his machine for tracing curves, is along similar lines. We have already referred (*1920*, 132) to the interesting illustrated account of this machine in the *Scientific American Supplement*, 1918, February 9 and 16 (partly reproduced in *La Nature*, 1919, September 27) —EDITOR.


1
2
3
4
5
6

and then

$$\pm 1 = \rho - 2 + \cos\theta,$$

so that the equation of the envelope becomes[2]

$$\rho = (1 - \cos\theta) + 2$$

and

$$\rho = 1 - \cos\theta.$$

There is therefore an inner envelope which is a cardioid, and an outer one in which the radius vector of this cardioid is increased by 2.

Fig. 1 has been drawn to represent these curves. This figure, like all those in this article, has the same position on the page that it had on the disk at the moment when it was completed and the drawing pen had returned to its initial position. If we can imagine the disk, which turns in a clockwise direction, to be now alone arrested, the pen will keep on moving up and down along the vertical line of the page through the cusp, that is, along what is generally denoted as the Y axis in figures, but which we may call here the mechanical axis of Y. The mathematical axis of $+ Y$ which is used in the equations just given and which convention directs to run always upward, runs to the right in this Fig. 1, so that the figure must be turned 90° in an anticlockwise direction in order to have it oriented in the usual way. The reason for this departure from the customary mathematical practice was that, by presenting the mechanical aspect of the figures, the changes that come over them when the initial phase or position of the pen or the rotation frequency of the disk is altered, may be seen to better advantage. The mathematical axes must therefore be rotated to suit each figure in particular. This will present no great difficulty.

The motion of the pen in Fig. 1 was the resultant of two simple harmonic movements both of the same amplitude, one with a unit period and the other with a period m, 15/16 or 16/15 as long, while the disk had a period of either component. In practice component A had a wheel with 32 cogs which made 15 revolutions while component B with 30 cogs made 16, the disk in the meantime with a 30- or 32-cog wheel making 16 or 15 turns. In Fig. 1 a 32-cog wheel with 15 revolutions was used on the disk. A radius (through the cusp) may be seen to cut the compound curve in 15 points. Had a 30-cog wheel with 16 revolutions been employed, there would have been 16 such intersections.

The pen was placed at the center of the disk (at the cusp) when both of its components were in phase 90°. When set in motion the pen started to draw a

[2] It is rather questionable to call m a variable parameter since only one value of m is considered at a time. There is a single curve with several lobes, giving the appearance of so many different curves all tangent to their envelope. Perhaps we might speak of it as the *envelope of the lobes*. It is true that this envelope happens to be the same for all values of m, at least for all rational values of m, and that its equation can be obtained by the usual process if we make m a variable parameter, but the envelope of a family of curves might be quite different from the "envelope" of the lobes of any one of them, and in the case of the curves represented by the given equation it might be difficult to apply the theory of envelopes to the variation of the given curve produced by a continuous variation of m.—EDITOR.

cardioid twice the size of the inner envelope, but this at once, although gradually, changed into a curve that became more and more curtate as the pen receded farther from the center at each revolution, until, in the middle of its compound period, when $\cos\theta + \cos m\theta$ was equal to zero, it momentarily drew the arc of a circle with the radius 2. After this the lobes of the curve repeated themselves in inverse order, while their axes kept on swinging in the same direction.

A study of Fig. 1 shows that the points of intersection of the lobes are arranged in radial lines at equal angular intervals, and that the points of tangency of the curve with the two envelopes are also spaced equiangularly.

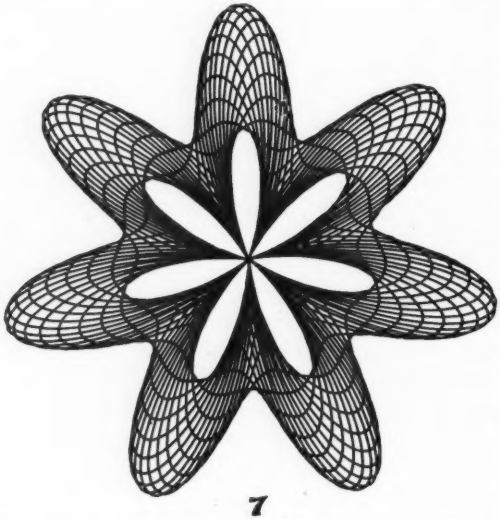
The Outer Envelope a Cardioid.—There is a second way of drawing an envelope that is a cardioid. In the first case we placed the pen at the center of the disk when the phase of each of its two harmonic components was 90°. Now let us make the phase 0° at the center. The first component A, if used alone, will then trace the circle $\rho = \sin\theta$, and the two together will trace $\rho = \sin\theta + \sin m\theta$. Proceeding as before, we find the envelope

$$\pm 1 = \rho - \sin\theta$$

or

$$\rho = 1 + \sin\theta$$

and $\rho = -1 + \sin\theta$ or $(1 + \sin\theta) - 2$, which are identical, or rather coincident, the first being traced as usual by the positive extremity of ρ, say by point P, and the other by a point P' at the constant distance of 2 from P in the negative direction of ρ. There is then practically only one envelope, which we may in the mechanical sense call an outer one. The mathematical axis of $+Y$ now runs to the left in Fig. 2. We may note that the equation of the second envelope of Fig. 2 has the constant -2 as opposed to $+2$ in the second one of Fig. 1.



7

Envelope Rosettes.—Generalizing the above results by using $n\theta$ in place of θ, we may apply the same principles to rosettes. Thus if we take $n=2$, there are two complete compound cycles of the pen to one of the disk, that is, the components A and B turn twice 15 and 16 times while the disk makes as before its usual 15 or 16 revolutions. We then have a rosette in the inner envelope of Fig. 3 and in the outer one of Fig. 4, using the latter expression in the mechanical sense. In the mathematical sense, however, there are two envelopes in Fig. 4, not coincident, but lying at

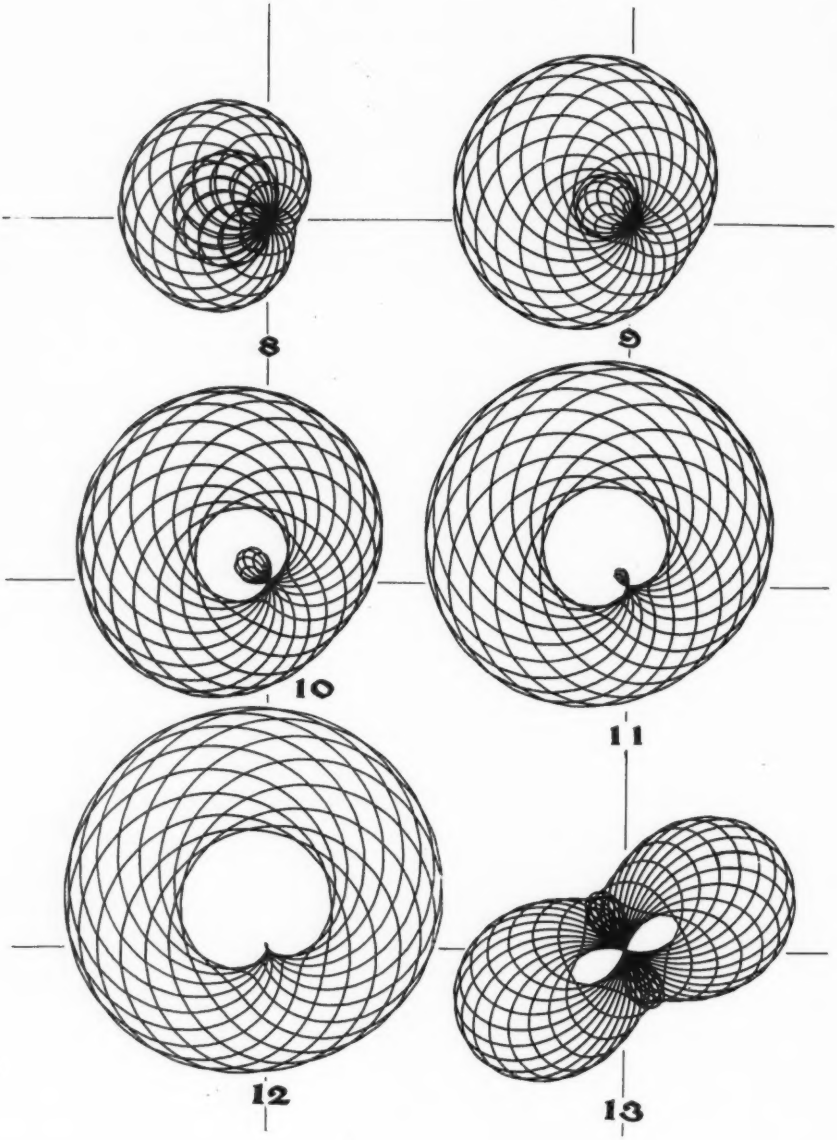
right angles to one another. The direction of the mathematical axes has also undergone a change. Their position in these and subsequent figures may readily be deduced from the respective equations, and will for that reason no longer be referred to.

When $n = 3$ we see that the usual inner envelope is a rosette in Fig. 5, and that the outer one in Fig. 6 is an equal one. Fig. 7 shows a septifolium as an inner envelope. The outer one was not drawn because it would have been almost totally black, as Fig. 6 leads us to suspect, on account of the great number of its close and overlapping lines.

From a mathematical point of view all these seven figures present envelopes that are rosettes. When the common phase of the two components A and B is made 90° at the center of the disk, as in Figs. 1, 3, 5, 7, we have two envelopes, an inner one which is a rosette, and an outer one in which the radius vector of the inner one is increased by 2. When the common phase is 0° at the center, as in Figs. 2, 4, 6, there are also two envelopes, both being equal rosettes. They are coincident when n is odd, but crossed equiangularly when n is even. From a mechanical standpoint, the first class of figures may be said to have rosettes as their inner envelopes and the second to have corresponding equal rosettes as their outer envelopes, the number of lobes being doubled in the latter case when n is even.

The Ratio of the Periods of the Components, or the Value of m.—In all the cases presented m was taken as 15/16 or 16/15, the harmonic component A making $15n$ revolutions and B $16n$, while the disk rotated 15 or 16 times. The number of revolutions of the disk, 15 or 16, must be $1/n$ that of one of the components. We may select either except when n is a factor of the one used, for then, as soon as the pen has run through one complete compound cycle, it will begin to retrace the curve already drawn, so that the figure will present a disappointing appearance of incompleteness, since it will have only one nth as many lines as it ought to have. For this reason the disk had to make 15 turns for $n = 2$ and 16 for $n = 3$. For $n = 7$, 15 were made, but 16 would have done equally well. For $n = 5$ (not shown) they had to be 16.

The Starting Phases of the Components.—When the phases of the components A and B were 90° and the pen was set down at the center of the disk, the inner envelopes it traced were the rosettes shown in Figs. 1, 3, 5, 7. The identical figures, only turned at right angles, were drawn when the pen was started in phase 0° on the mechanical Y axis at the distance + 2 from the center, that is, at the upper end of a lobe. The reason is that in a quarter of a turn of the disk one of the components A or B advances exactly one or n quarters of a period also and the other only one-fourth of 1/15 or 1/16 more or less. This difference is insensible in practice when $m - 1$ is very small. In like manner the identical "outer-envelope" rosettes, turned at right angles, resulted, Figs. 2, 4, 6, when the pen was started on the Y axis at the distance + 2 in phase 90° instead of at the center in phase 0°. From this it follows that the pen may be started in any equal phase α of its components and set down on the mechanical Y axis at the


8
9
10
11
12
13

distance 2 sin α from the center in the last case and 2 cos α in the first, in order to draw the same respective rosette, which will then be turned through the angle α on the disk. For this reason we might call the rosettes in Figs. 1, 3, 5, 7, "cosine" rosettes and those in Figs. 2, 4, 6, "sine" rosettes.

Transition Envelopes.—The principle just stated may be applied to show the transition from the (mechanical) outer- to the inner-envelope cardioid. Thus Figs. 8–12 are intermediate between Figs. 2 and 1. In all of these seven figures the starting point was at the center of the disk, but the phases were taken at 15° intervals from 0° to 90°. In Fig. 2 the phases of the pen at the start were 0°. In Fig. 8 the phases were 15°, in Fig. 9, 30°, in Fig. 10, 45°, in Fig. 11, 60°, in Fig. 12 75°, and finally in Fig. 1, 90°. The transition may thus be readily followed, and the axis of the envelope seen to swing round with uniform speed.

This identical series of seven transition envelopes might have been obtained by keeping the starting phases of the components at 90° and setting down the pen on the mechanical Y axis at the distance of twice the sines of 0°, 15°, $\cdots$ 90° from the center. In this case the axes of the envelopes would have remained stationary on the mechanical X axis. When the pen is set beyond the distance 2 sin 90°, the envelopes become curtate. Their inner faces will be the outer ones in Figs. 2, 8–12, 1, while their outer ones will tend to become more circular.

Unequal Starting Phases of the Components.—Instead of starting the pen with its components in equal phases, phase differences of any magnitude may be used. By studying the usual generation of Fig. 1 as given before, the initial position of the pen on the mechanical Y axis may so readily be deduced from the position it has there corresponding to the given phase difference, that numerical exemplifications are not necessary. The application to rosettes in general is also sufficiently obvious.

Finally Fig. 13 shows a transition envelope for an even value of n intermediate between Figs. 3 and 4.

THE COLLEGE AS A TRAINING SCHOOL FOR HIGH SCHOOL TEACHERS.[1]

By ERNEST B. LYTLE, University of Illinois.

This audience is no doubt familiar with the wonderful growth of high schools in the United States. In 1906 there were 52,394 pupils enrolled in the high schools of Illinois; in 1916 there were 102,870 enrolled, an increase of 96 per cent in ten years in Illinois alone. (*Illinois High Schools*, L. W. Smith, p. 9.) In the light of the recent marvellous growth in the number and community importance of our high schools, it is not necessary to argue here the great significance

[1] Address read before the Ann Arbor meeting of the Mathematical Association of America, Sept. 5, 1919; also before the Illinois Section of the Association, Nov. 22, 1919.

of training high school teachers. Colleges must see the problem and attack it vigorously. In training able teachers our colleges and universities not only render the high schools and their communities a great service but they help themselves by improving the preparation of the students coming to them for further education.

Our particular present interest is to discuss the part mathematicians are now taking in teacher preparation and to suggest ways and means of increasing their service in this direction. Good teaching requires (1) knowledge of subject matter and (2) a sympathetic understanding of students. There is no upper limit to the amount of scholarship desirable, the only question is how much scholarship is possible under given conditions. The few people who believe one can know too much to teach well are looking in the wrong place for the trouble; it is not too much scholarship but rather too little sympathetic understanding of students which makes many failures in teaching. However true it may be that some great scholars are poor teachers because they have permitted themselves to grow unsympathetic and impatient with immature minds, yet sympathetic understanding of the difficulties of less mature persons is by no means incompatible with high scholarship; we all know many master teachers who possess both scholarship and sympathy for the learner.

While recognizing no upper bound to knowledge of subject-matter desired in a teacher, it is still pertinent to inquire, "How much mathematics is it reasonable to require of prospective teachers of mathematics under present conditions?" Today our colleges are quite generally requiring from 20 hours (one hour a day for two school years) to 30 hours (one hour a day for three years) in mathematics to obtain their official recommendation to teach in our high schools. In many cases prospective teachers elect more than the minimum requirement from the regular advanced courses in mathematics. The usual sequence of courses through the calculus is quite uniformly required; but beyond this there seems to be little absolute requirement and considerable variation in practice. Most colleges making any pretense at high school teacher training offer a special Teachers' Course which considers the values, methods, subject matter, texts, reform movements, and best literature of secondary mathematics. The University of Michigan, in 1893, was the first college to offer a course on the teaching of algebra and geometry and by 1912 there were 34 American colleges offering such courses (Bibliography 2, p. 6). In 1916 there were 40 out of 100 selected colleges and universities offering some such teachers' course (Bibliography 8, p. 395). When offered such a course is required, or very strongly advised, for recommendation to high school mathematics positions. Most students elect a course in the theory of equations and determinants as an advanced course in algebra; not quite so uniformly they elect a course in projective or modern geometry. You are no doubt familiar with the discussions in the MONTHLY on the value, methods and content of such an advanced course in geometry.[1] As brought out in these discussions the value and importance of such a course in modern geometry is not

[1] AM. MATH. MONTHLY, Jan., 1914, p. 30, 31, 32; Feb., 1914, p. 63; Apr., 1918, p. 159.

yet fully appreciated. History shows how for a period the world neglected synthetic methods in its enthusiasm over the newly discovered analytic geometry, but later had to be brought back to a realization of the great beauty, power, and simplicity of synthetic geometry methods; and so also teachers-to-be after being introduced to the wonderful analytic geometry, in a later course need to be impressed with the value of a combination of the two methods.

Another important course is a unifying course generally offered under some such title as "Fundamental Concepts" or "Synoptic Course." Teachers particularly need to have perspectives, foundations, and essential conceptions impressed upon them. Teachers in high schools have spoken to me with considerable enthusiasm on the value of such a course. Young's book on *Fundamental Concepts of Algebra and Geometry* has ably made a start in this direction but in my own opinion we mathematicians are not yet appreciating and emphasizing this type of work as strongly as it deserves. Mitchell found only 8 such courses in the 100 institutions which he investigated (Bibliography 8, p. 396).

Teachers need work in the History of Elementary Mathematics since historical remarks in the class-room arouse interest; and teachers familiar with the historical background of their subject are not so easily misled by radical would-be reformers. History courses were offered in 24 of Mitchell's 100 institutions; 9 others devoted part of their teachers' course to history, making 33 institutions giving some formal teaching in the history of mathematics (Bibliography 8, p. 396). Columbia University and the University of Michigan offer graduate courses in the History of Mathematics.

These mathematics courses offered particularly for students preparing to teach—"Teachers' Course," "Fundamental Concepts" or "Synoptic Course," and "History of Elementary Mathematics"—are usually taken in the senior, occasionally in the junior year, and with few exceptions require as prerequisites courses at least through the calculus. Sometimes these courses are offered in the Schools of Education but generally by instructors specially trained in pure mathematics. Teachers College of Columbia University offers the widest variety of courses on the teaching of mathematics; their 1919–1920 special bulletin offers 11 different courses not counting practice teaching; the 1917–18 catalog of the University of Chicago School of Education offers four such courses and the University of Illinois gives three special teachers' courses.

One type of valuable teacher training seems to be nowhere explicitly offered, that is, the oral presentation of papers on special topics worked up in the libraries. Teachers certainly need training in the use of libraries, acquaintance with books other than text-books, as well as practice in the lucid and convincing presentation of well-organized material collected from the best sources our libraries afford. No doubt many colleges give some of this type of training implicitly in connection with other courses or in clubs. We raise the question whether this important type of training is not being neglected by being allowed to remain implicit? Is not a seminar for undergraduates aimed explicitly to train in the use of libraries and oral presentation worthy of emphasis? Is it not probable that high school teachers

thus definitely introduced to the riches of our libraries would show a greater interest in mathematics as a science and make greater efforts to increase their knowledge of the subject by more individual reading and study?

Besides the mathematics courses specially considered students preparing to teach frequently elect even up to a total of 40 hours from courses in pure mathematics such as differential equations, theory of functions of both real and complex variables, differential geometry, solid analytic geometry, higher algebra, limits and series. They sometimes elect applied courses in statistics, astronomy, physics and mechanics.

When so many colleges require twenty or more hours in mathematics before they will recommend graduates for high school positions, why is it that we still find so many of much less training teaching mathematics today? One great cause is the practice, which administrators believe is a necessity, of assigning to a teacher well prepared to teach one or two subjects classes in a third or fourth and sometimes a fifth subject. In our high schools we find many teaching mathematics as a third or fourth subject who make no pretense of either special training or interest in mathematics. One suggestion for meeting this so called necessity is to increase the number of junior, township and community high schools since these types of organizations make possible earlier and more complete departmentalization and make places for more special teachers who can devote their time to one or at most two departments of instruction.

Up to this point, with the possible exception of the "Teachers' Course," we have considered only that part of teacher training which aims at knowledge of subject matter. But mathematics in our high schools is a means of educating boys and girls; student psychology is the second and a very important side of teacher training. "Professional Training" is the term usually employed to characterize that part of teacher training which aims to give knowledge of adolescent interests, motives, and learning processes. A teacher may be saturated with mathematical knowledge and yet find it impossible to induce lively youngsters to increase their knowledge of the subject. The fluent character and great variations in adolescent mental life make teaching an arduous task. Too many pass by this side of teacher training with the dogmatism, "Oh, teachers are born, not made," or "Actual experience is the only way of learning how purposefully to influence students." Is it not as reasonable to say that physicians and surgeons must get all their knowledge of the human body from practice? Doctors are required to spend long periods in the study of human physiology and anatomy with all the generalizations from the experiences of previous successful doctors before they are permitted by law to practice medicine themselves. Why should it be thought unreasonable to expect teachers to study the experience of successful teachers and get their generalizations and suggestions before they are permitted to undertake the important work of directing the mental life of our young people? While granting all possible importance to natural abilities yet in no work at all comparable with education in importance do we rely entirely on methods of trial and error in experience with no preliminary training in theory. Notwithstanding

some objectors the requirement of some professional training for teachers is already a fact. The North Central Association of Colleges and Secondary Schools, including over one thousand high schools, now requires that "the minimum professional training of teachers of any academic subject shall be at least eleven semester hours in education" (Bibliography 7). The Illinois school law now requires an examination in educational psychology, and the principles and methods of teaching for a state high school teacher's certificate (*1919 Illinois School Law*, Circular 138, p. 6).

Schools and Colleges of Education are meeting this demand by developing courses in educational psychology, principles of secondary education, theory of teaching, special methods in particular subjects, observation and practice teaching, history of education, and educational sociology. While we who are workers in an old, exact and well-developed science may have felt that some of the past work in education has been unscientific in method and contains hasty and unwarranted generalizations, yet is not blundering always common in a new field? Education is very very much younger than mathematics and yet educational investigators of really scientific ideals have already appeared. Cannot we mathematicians be constructive rather than merely destructive critics; can we not be sympathetic and suggestive in our attitude toward this comparatively new, important and complex field of investigation? A body of educational facts and guiding principles is slowly being accumulated to help the teacher. The wide use of mental tests during the World War is a tribute to the painstaking workers in education. We should therefore require prospective teachers to take good courses in educational psychology, technique and theory of teaching, history of education, and, in case they have never taught, some practice teaching as their work in Professional Training.

The work in practice teaching is being constantly improved in quality and there is a noticeable tendency to raise the standards of scholarship of supervisors of practice teaching in our demonstration high schools. For example, at the University of Illinois where a new high school is about to be opened, the ideal set for the supervisor of mathematics work is one with a Ph.D. degree in both mathematics and education, if such cannot be found then there must be a Ph.D. degree in mathematics and as much training in education as possible. Likewise in each department they plan to require a doctor's degree in the field to be supervised together with specialization in education.

The Wisconsin plan of "directed teaching" in contrast to practice teaching deserves special mention. This plan is fully described in the *Eighteenth Year Book of the National Society for the Study of Education*. Limited time makes possible here only a hasty sketch of this unique plan. Each class is at all times in charge of an expert regular staff teacher; from one to three college seniors in training are assigned to a particular class; they become students in this class and participate in all class activities by preparing lessons and reciting just as regular students do, and this participation gives them the exact student viewpoint; at the same time these seniors discuss methods of organization and technique with the

expert teacher in charge to get the teacher's viewpoint; by consistent unfailing excellence a senior may win the right to assist the staff teacher and finally becomes the class leader for short or even long periods, but always under the direction of the staff expert; preparation through participation is the key note of this plan. The purpose of this special mention of the Wisconsin directed teaching scheme is its suggestions. Why cannot this plan be adapted to teacher training in our college classes as now organized? Why not assign one prospective teacher to each expert member of our present college faculties; by participation, assisting and personal contact with this expert teacher cannot the student get the best possible teacher training? Would not a plan of associating inexperienced beginners with older experts for careful direction and help be much better than our present plan of putting assistants in sole charge of sections to sink or swin unaided? Can we get practice teaching in college classes without waiting for the formation of practice schools which are often long delayed by the budget difficulties? It is not suggested that experimental schools be abolished or discouraged but rather that teacher training possibilities may be extended to college classes as at present organized.

Besides furnishing college students with first class teacher training our colleges and universities should also help teachers already in service. College instructors should coöperate heartily in high school teacher associations. The annual University of Illinois High School Conference has been a very potent factor in teacher contact and mutual inspiration; it has given high school teachers the points of view and advice of more expert mathematicians and it has given college instructors more sympathetic understanding of the high school teachers particular problems. The teachers' book-shelf should also receive the careful attention of college men. There is need for books in English such as Klein's *Elementarmathematik vom höheren Standpunkte aus.* High school teachers need more books like Young's *Fundamental Concepts of Algebra and Geometry,* Carson's *Mathematical Education,* Nunn's *Teaching of Algebra,* Whitehead's *Introduction to Mathematics* and *Organization of Thought,* Keyser's *Human Worth of Rigorous Thinking, Monographs on Mathematics* edited by Young, and specially little contributions similar to Teubner's "Mathematische Bibliothek" on topics like *Der Begriff der Zahl in seiner logischen und historischen Entwicklung, Der Pythagoreische Lehrsatz, Konstruktionen in begrenzter Ebenen,* etc. The writing or translating of books which will find places on the secondary teachers' book shelf would be a distinct service in raising standards of scholarship among teachers of mathematics.

In order that the important work of secondary teacher training should not be neglected each college department of mathematics, specially the larger ones, should assign to one member of its staff the problem of elementary teacher training. This instructor should visit high schools as often as possible and keep in touch with actual high school conditions; he should take as his field of research the problem of devising ways and means of improving instruction in mathematics.

Summing up the suggestions as points for your discussion, we college teachers

can help high school teacher training (1) by emphasizing the desirability of the greatest possible scholarship in mathematics, (2) by developing strong courses in theory of equations, advanced geometry, fundamental concepts and history, (3) by emphasizing in all our courses perspectives and big ideas, (4) by developing the use of libraries, knowledge of best literature and ability to prepare and forcefully present papers on mathematical topics, (5) by making one member of the staff responsible for investigation in the field of teacher training, (6) by encouraging professional training specially in adolescent psychology, (7) by taking constructive and not destructive attitudes toward scholarly investigation in education, (8) by demanding that supervisors of mathematics teacher training work have a Ph.D. or equivalent in mathematics as well as special training in education, (9) by studying the Wisconsin directed teaching plan with the purpose of modifying it to apply to college classes, and (10) by writing and translating books or articles which appeal to the present interests and attainments of high school teachers.

American standards of mathematics teacher preparation, both in scholarship and professional training, are below those of some other countries, specially those of France and Germany, and we are still far from the ideal set up by our American Commission on Mathematics Teaching (Bibliography 2, pp. 13–14). However American standards are slowly but surely rising; let us help this upward movement.

BIBLIOGRAPHY.

1. *The Training of Teachers of Mathematics*, by R. C. Archibald. Bulletin, 1917, No. 27, U. S. Bureau of Education.
2. *Training of Teachers of Elementary and Secondary Mathematics.* Bulletin, 1911, No. 12, U. S. Bureau of Education.
3. *Influences Tending to Improve the Work of the Teacher of Mathematics.* Bulletin, 1912, No. 13, U. S. Bureau of Education.
4. "The Training of Mathematics Teachers in the Secondary Schools of the United States" by A. W. Stamper, *School Science and Mathematics*, April, 1910.
5. "The Professional Preparation of High School Teachers." *Eighteenth Yearbook of the National Society for the Study of Education*, 1919. This has a bibliography of 77 titles on teacher training.
6. "Report of the Committee of Seventeen on the Professional Preparation of High School Teachers." *National Educational Assoc. Proceedings* for 1907.
7. *1919 Report of North Central Assoc. of Colleges and Secondary Schools.*
8. "What are the actual courses now offered in colleges and universities in this country for the preparation of teachers?" by U. G. Mitchell and R. C. Archibald. AM. MATHEMATICAL MONTHLY, Dec., 1916, pp. 395–399.
9. "Geometry for Juniors and Seniors" by E. B. Stouffer, AM. MATH. MONTHLY, April, 1918.
10. "Means for Scientific Development of Mathematics Teachers," by G. A. Miller, *Science*, Dec. 6, 1918.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

In our last number, Professor Rees gave an instance of the use of vectors in connection with a problem in dynamics. In the first discussion below he indicates how the vector notation may be used to attack several questions in elementary geometry. It cannot be said that the proofs are much shorter than the usual ones; but the elegance of the vector formulation is obvious.

In an article in this department last October, devoted to the application of the theory of probability to various questions connected with the game of "craps," Mr. B. H. Brown gave the following theorem: *In any series of games where the probability of winning is constantly p, the average number of games won, up to and including the first lost game, is the reciprocal of the probability of losing.* He used this theorem to determine the average number of "rolls" required to complete a game. His proof, although relatively simple, involved the summation of the power series for $(1 - p)^{-2}$. In the second discussion this month Miss Charlotte Dickson gives an immediately obvious proof of the theorem. While the wording is phrased with reference to Mr. Brown's particular application, the work clearly holds for the general theorem.

Professor Frumveller contributes an "experience" article on the teaching of logarithms, suggested by the paper of Professor McClenon in the September MONTHLY. Like many articles on methods of teaching, its value lies not so much in the presentation of new pedagogical ideas as in recommendations concerning the distribution of emphasis. All teachers who are not committed to pure formalism will agree that equivalence of the logarithmic and exponential relationships, $y = \log_b x$, $x = b^y$ must be presented as the very soul of the theory of logarithms. With regard to Professor Frumveller's discussion of the difficulties associated with the negative characteristic, it is fair to state that none of the objections rightfully made to the notation -3.93817 can be extended to the notation $\bar{3}.93817$, which may legitimately be understood to mean $-3+0.93817$. Professor Frumveller closes his discussion by answering the query, for this kind of problem, when the wrong method can give the right result. The answer involves an elementary notion of number-theory.

Mr. C. N. Schmall in the closing discussion gives an illustration of the study of geometric questions by analytic methods. Sometimes the view is expressed that Cartesian coördinates should be introduced into the course in geometry in the secondary schools—after proofs of theorems on parallels and perpendiculars have made this logically possible—and used for certain demonstrations in lieu of the Euclidean tools. If such a procedure is to be seriously considered, it is desirable to produce instances in which analytic proofs are preferable. Can our readers supply a few such cases?

I. Theorem Concerning the Concurrency of Three Lines.

By E. L. Rees, University of Kentucky.

The theorem stating the condition for the collinearity of three points given by their position vectors, is well known to students of vector analysis.[1] It is the purpose of this note to state and prove a similar theorem concerning the concurrency of three lines and to illustrate how this theorem may be used.

Theorem. *The necessary and sufficient condition that the lines AA', BB', and CC' be concurrent is that the position vectors of the points determining the lines satisfy equations of the form*

$$a\boldsymbol{A} + a'\boldsymbol{A}' = b\boldsymbol{B} + b'\boldsymbol{B}' = c\boldsymbol{C} + c'\boldsymbol{C}'$$

the scalar coefficients of which satisfy the relations

$$a + a' = b + b' = c + c'.$$[2]

Proof. It is a necessary condition; for if the lines are concurrent the position vector of the point of intersection may be expressed in terms of each pair of vectors as follows:

$$\boldsymbol{P} = \frac{a_1\boldsymbol{A} + a_1'\boldsymbol{A}'}{a_1 + a_1'} = \frac{b_1\boldsymbol{B} + b_1'\boldsymbol{B}'}{b_1 + b_1'} = \frac{c_1\boldsymbol{C} + c_1'\boldsymbol{C}'}{c_1 + c_1'}$$

and since the sum of the scalar coefficients of each member is unity the condition is satisfied. The reader can easily see that the condition is also satisfied when the lines are parallel or coincident.

It is a sufficient condition; for if we divide member by member we have

$$\frac{a\boldsymbol{A} + a'\boldsymbol{A}'}{a + a'} = \frac{b\boldsymbol{B} + b'\boldsymbol{B}'}{b + b'} = \frac{c\boldsymbol{C} + c'\boldsymbol{C}'}{c + c'},$$

each member of which is a vector terminating in one of the lines, and since these vectors are equal they must terminate in the same point and the lines are therefore concurrent. It is assumed in the proof that the members of the second set of equations are not zero. If they are zero the lines are parallel or coincident.

Very brief and simple proofs of many of the theorems of geometry involving the concurrency of three lines result from the application of this theorem. As illustrations we prove the following.

Desargues's Theorem. *If two triangles are so situated that the lines joining corresponding vertices are concurrent the corresponding sides intersect in collinear points.*

Proof. If we denote the two triangles by ABC and $A'B'C'$ then by the above theorem we have for hypothesis

[1] Cf. Gibbs-Wilson, *Vector Analysis*, New York, 1909, p. 31.—Editor.

[2] We shall understand that the term concurrent as used here includes the special cases for which the lines are parallel or coincident.

$$\begin{cases} a\boldsymbol{A} + a'\boldsymbol{A}' = b\boldsymbol{B} + b'\boldsymbol{B}' = c\boldsymbol{C} + c'\boldsymbol{C}', \\ a + a' = b + b' = c + c'. \end{cases}$$

Calling the intersections of AB, BC, and CA with the corresponding sides of the other triangle, D, E, and F respectively, and proceeding in the usual way we find[1]

$$(a - b)\boldsymbol{D} = a\boldsymbol{A} - b\boldsymbol{B},$$

$$(b - c)\boldsymbol{E} = b\boldsymbol{B} - c\boldsymbol{C},$$

$$(c - a)\boldsymbol{F} = c\boldsymbol{C} - a\boldsymbol{A}.$$

Adding we have

$$(a - b)\boldsymbol{D} + (b - c)\boldsymbol{E} + (c - a)\boldsymbol{F} = 0.$$

Since the sum of the scalar coefficients of this equation is zero the theorem is proved.

It will be noted that the triangles may be in the same or different planes. Thus the proof is perfectly general.

CEVA'S THEOREM. *If in the triangle ABC the points A', B', and C' are taken on the sides BC, CA, and AB respectively, in such a way that*

$$(AC'/C'B)\cdot(BA'/A'C)\cdot(CB'/B'A) = 1,$$

then the lines AA', BB', and CC' are concurrent.

Proof. Calling the first and second ratios s/r and t/s respectively, the third ratio must be r/t by the hypothesis of the theorem. We have then

$$(s + t)\boldsymbol{A}' = s\boldsymbol{B} + t\boldsymbol{C},$$

$$(t + r)\boldsymbol{B}' = t\boldsymbol{C} + r\boldsymbol{A},$$

$$(r + s)\boldsymbol{C}' = r\boldsymbol{A} + s\boldsymbol{B}.$$

Adding $r\boldsymbol{A}$ to the first equation, $s\boldsymbol{B}$ to the second, and $t\boldsymbol{C}$ to the third, we see that the second members of the resulting equations are equal. Hence

$$(s + t)\boldsymbol{A}' + r\boldsymbol{A} = (t + r)\boldsymbol{B}' + s\boldsymbol{B} = (r + s)\boldsymbol{C}' + t\boldsymbol{C}.$$

The scalar coefficients of these equations satisfy the condition of the theorem, hence the proof is complete.

II. ON A THEOREM IN THE THEORY OF PROBABILITIES.

BY CHARLOTTE DICKSON, American Telephone and Telegraph Co., New York.

In this MONTHLY for October, 1919, Mr. Bancroft H. Brown gives a method for determining the "average number of rolls needed to decide a game of craps."

[1] We assume here that $a - b$, $b - c$, and $c - a$ are not zero. If any of these differences vanish then some of the corresponding sides are parallel, *i.e.*, intersect at infinity. Our justification for extending the proof to include the points at infinity lies in the fact that the points of intersection remain collinear when one or more of them recede to infinity.

A solution of the same problem by a somewhat simpler method might be of interest.

Let X be the average number of "rolls" needed further to decide a "point." Now we know that it is necessary to make at least one roll; and the probability of this, being certainty, is expressed by unity. If the "point" is not decided at the first roll (the probability of which is p or $1 - q$), the average number of rolls needed is still X. Therefore we arrive at the following equality,

$$X = 1 \times 1 + (1 - q)X$$

from which we find that $X = 1/q$.

This solution and that suggested by Mr. Brown are given by Louis Bachelier in his *Calcul des Probabilités*, volume 1, Paris, 1912, page 11.

III. Concerning the Teaching of Logarithms.

By A. F. Frumveller, Marquette University.

In this Monthly for September, 1919, Professor McClenon brought up the question of introducing logarithms by the historic method of geometric progression; he mentions as an alternative, the complete abandonment of all theory, and the laying down of mechanical rules of thumb for the student until the use of the tables has become automatic. The first or historic method has, it seems, been tried out successfully at the Hyde Park High School, Chicago; Mr. Josef Nyberg explained his proceedure in the October number of this Monthly for 1918, p. 337, but on reading his explanation it would seem as if the lack of directness of this approach would render it unsatisfactory in the hands of teachers less skillful and inspiring than Mr. Nyberg himself.

The late S. A. T. C. arrangement gave the present writer a splendid opportunity of trying out on a large scale the teaching of logarithms to students of average ability; and it so happened that the few days spent in this study afforded the only agreeable interlude in an experience that to most of us was like a nightmare. Our method was this; an equation like $y = x^2$ is written down; we define 2 to be the logarithm of y to the base x, and write $\log_x y = 2$ as the statement of this definition. This we call "translating the exponential equation to the logarithmic form," and the student is drilled at once in making translations of this kind by the dozens, till he can do it automatically; numerical cases like $9 = 3^2$ are next handled orally, with all possible modifications; *e.g.*, if 3 is the log of 8, what base am I using? etc. Many algebras are well supplied with such oral exercises; after a lively quiz of this sort, the students, *all* of them, had the idea of a logarithm firmly fixed in their minds. The rules for operating with logarithms gave no difficulty, since by definition a logarithm is merely another name for an exponent; fractional and negative exponents were then brought in and *translated* into logarithmic form, using always an exponential equation $y = x^{*n}$ as a starting point, and then employing numbers in place of y, x, and n, till the idea took root.

This plan worked excellently; the students learned rapidly how to operate this new notation; as regards the tables, they were merely told, that just as in banks the computation of interest is never done by the schoolboy's rules of arithmetic, but by means of tables which have been computed in advance, so in mathematics, the exponents of all possible numbers have been computed for the base 10 to save us trouble; the students are to feel free to use these tables. We always spoke of the tables as giving *exponents*; "look in the table of exponents" was one of our standard remarks.

Fractional exponents like 1/3 will be found in the tables in decimal notation, as 0.33333; to make this usage familiar, we had only to take our old equation $y = x^n$ and replace n by some terminating decimal like 0.25 or 0.125 to begin with, and then go back to the fractional form and interpret what we had. In regard to decimals, we defined that if a number like 7.25 is given, we call 7 the characteristic, and 25 the mantissa *of this number*. And right here we met the first and the only difficulty! Some of the students had used negative characteristics in their former work, writing for instance -3.93817 as the log of .008673; the idea was to get the sixth root of this number. On dividing by 6, we got $-.65636$! What was the matter with this result? I recommended that we write $3.93817 - 6$ before division, which gives $0.65936 - 1$, correctly; but at once some one wanted to know why we could not write $7.93817 - 10$ just as well, the result being apparently altogether different? There was no need of assigning this topic for the next recitation! The way in which we finally threshed the matter out was this.

The decimal notation .008673 is merely a shorthand notation for the proper fraction $\frac{8,673}{1,000,000}$ whose log is $3.93817 - 6$; in every case therefore, the log consists of two parts, a positive part for the numerator, and a negative part for the denominator. In taking the sixth root, we must operate on both numerator and denominator; hence 6 is to be used as a divisor on both parts of the logarithm. Moreover there is not, and cannot be, any other correct form for the writing of the logarithm of a decimal; the log is essentially composed of two parts, the first of which is integral and decimal, the second integral only, since it represents some power of ten. Any other way of writing such logs is essentially wrong and misleading; I can transform my logarithm by adding to it zero in the shape of $m - m$; I can actually perform the subtraction indicated between the two parts and write $3.93817 - 6 = -2.06183$; but I cannot reasonably or intelligently keep the decimal unchanged and merely subtract the integers from one another, so as to have $3.93817 - 6 = -3.93817$! What would any one say if I were asked to subtract 4 from $3\frac{1}{7}$ and I were to write the result thus: $3.14285 - 4 = -1.14285$? Yet this foolishness is what many writers recommend, and no wonder the student is mystified; a number like 3.93817 is one definite mental object, and when a minus sign precedes it, this minus sign always refers to the entire number, and not to a certain percentage of the digits that compose it! Consider the following variety of ways for finding $\sqrt[6]{0.008673}$:

(1) $\dfrac{6)3.93817 - 6}{0.65636 - 1}$, (2) $\dfrac{6)57.93817 - 60}{9.65636 - 10}$,

$n = (4.5329)/10 = 0.45329$; $n = (4532900000)/10{,}000{,}000{,}000 = 0.45329$;

(3) $\dfrac{6)7.93817 - 10}{1.32303 - 1.66666}$, (4) $\dfrac{6)2.93817 - 5.00000}{0.48969 - 0.83333}$,

$n = (2.104)/(4.641) = 0.4533$; $n = (3.088)/(6.813) = 0.4533$;

(5) $\dfrac{6)- 2.06183}{- 0.34364}$, (6) $\dfrac{6)- 3.93817}{- 0.65636}$,

$n = 1/(2.206) = 0.4533$; $m = 1/(4.5329) = 0.2206$.

The first two are the standard methods, but all are correct, except the last; here by a *mere accident*, the digits of the quotient turn out to be right, though the negative sign throws the number into the denominator; had we taken the seventh root, not even the digits would have been found, much less the proper sign!

After studying these results, the class voted unanimously that the use of negative characteristics was absurd and misleading; every one of these boys could use logarithms easily and correctly from that time forward. As an experiment in teaching logarithms, complete success was obtained, so that the writer sees no need of ever changing this method.

As a matter of curiosity, the question of settling under what circumstances the digits of the mantissa (in case no. 6 above) turn out correct was looked into; the reader may be interested in the answer. Let us use 0_k as a symbol for k zeros adjacent to one another in a given decimal number N; and let

$$\begin{aligned}\log N^x &= \log (0.0_k n_1 \cdots n_\lambda)^x \\ &= x[\log (n_1 \cdots n_\lambda) - \log (10^{k+\lambda})] \\ &= x[(\lambda - 1).s_1 \cdots s_5 - (k + \lambda)] \text{ as correctly written.}\end{aligned}$$

Written *wrongly*, by subtracting the integral parts of these numbers while leaving the decimal part untouched, we get

$$\log (N^x) = x[- (k + 1) \cdot (s_1 \cdots s_5)].$$

Now let $x = 1/y$; then if $|k + 1| \equiv (\lambda - 1)$, (mod y), *i.e.*, if the quotients $(k + 1)/y$, $(\lambda - 1)/y$, have the same remainders, the mantissa $(s_1' \cdots s_5')$ in both cases will be the same.

IV. Relating to the Analytical Geometry of the Circle.

By Charles N. Schmall, New York.

E. H. Askwith in his *Analytical Geometry of the Conic Sections* (London, A. & C. Black, 1908), p. 78, § 90, says: "We could by analysis prove all the geometrical properties of the circle. It must not, however, be supposed that

these properties are in all cases more easily proved by analysis than by pure geometry. Sometimes the methods of analysis are short and simple; but there are cases where they are complicated and inferior to the methods of pure geometry." He then proceeds to give a case illustrating the latter part of his statement, taking an example in which the use of analysis offers no advantage. He proves analytically that angles in the same segment of a circle are equal.

As an illustration of the alternative case of Mr. Askwith's remarks, I submit the following discussion, by analysis, of the relation between two circles in a plane. This example may prove interesting to students, as an instance where analytical methods are useful.

Let R and r be the radii of the two circles, and d the distance between their centers. Take the center of the first circle as the origin of rectangular coördinates, and the line through the centers as the x-axis.

Then the equations of the two circles are

$$x^2 + y^2 = R^2, \tag{1}$$

$$(x - d)^2 + y^2 = r^2. \tag{2}$$

Solving (1) and (2) for x and y, we get

$$x = \frac{R^2 - r^2 + d^2}{2d}, \tag{3}$$

$$y = \pm \frac{1}{2d} \sqrt{4R^2d^2 - (R^2 - r^2 + d^2)^2}$$

$$= \pm \frac{1}{2d} \sqrt{(2Rd + R^2 - r^2 + d^2)(2Rd - R^2 + r^2 - d^2)}$$

$$= \pm \frac{1}{2d} \sqrt{\{(R + d)^2 - r^2\}\{r^2 - (R - d)^2\}}$$

$$= \pm \frac{1}{2d} \sqrt{(R + d + r)(R + d - r)(r + R - d)(r - R + d)}$$

or,

$$y = \pm \frac{1}{2d} \sqrt{(R + r + d)(R - r + d)(R + r - d)(r - R + d)}. \tag{4}$$

An examination of (3) and (4) shows that:

1. The abscissa x is always real.

2. If the continued product under the radical sign in (4) is *positive*, then the two points of intersection of the circles are *real*. These two points have the same abscissa, and two ordinates equal, but of opposite signs. Hence, *when two circles intersect, their common chord is bisected perpendicularly by the line through their centers.*

3. Since the first factor under the radical sign is always *positive*, the two values of y will be *real* if (*a*) the three other factors are *positive*, or (*b*) one of them is

positive and the remaining two *negative*. But condition (*b*) cannot exist; for if one of the three said factors be *negative*, the remaining two must necessarily be *positive*. Thus, supposing the second factor, for example, to be *negative*, then we have

$$R - r + d < 0;$$

$$\therefore \quad R + d < r;$$

hence

(5) $$d < r,$$

and

(6) $$R < r.$$

Hence (5) shows that the *third* factor must be positive and (6) shows that the *fourth* factor is positive. Similarly, if either the third or fourth factor be supposed negative the remaining two can be shown to be positive. Furthermore, the three stated factors cannot all be negative together. Hence, the only supposition tenable (for both values of y to be real) is (*a*) above; *i.e., all four factors must be positive.* Therefore:

4. *Two circles intersect in two real points when the sum of each two of the three quantities R, r, d, is greater than the third.*

5. If any one of the three said factors be *zero*, we get $y = 0$, and the two circumferences have only one common point, which lies on the x-axis; *i.e., the point of contact of two tangent circles lies on the line through the centers.*

6. The first factor, obviously, cannot vanish. Hence, if the two circles are to have only one point in common, *i.e.*, are to touch each other, then the *second, third* or *fourth* factor must vanish.

If the second or fourth factor vanishes,

$$R - r = \pm d,$$

or

(7) $$d = |R - r|.$$

If the third factor vanishes,

(8) $$d = R + r.$$

Therefore:

7. *Two circles will touch each other when the distance between their centers is equal to the sum or difference of their radii.*

8. If any one of the three factors mentioned be negative, the remaining two are positive, by 3. In that case the values of y will be *imaginary*; *i.e.*, the circumferences will not meet. Hence:

9. The circumferences will not meet if

(9) $$R - r + d < 0, \text{ i.e., } r - R > d,$$

or,

(10) $$R + r - d < 0, \text{ i.e., } R + r < d,$$

or,

(11) $$r - R + d < 0, \text{ i.e., } R - r > d.$$

Therefore, by (9) and (11), the circumferences will not meet when the difference of the radii is greater than the distance between the centers, *i.e.*, in that case one circle lies wholly within the other. Also, by (10), the circles will not meet when the sum of the radii is less than the distance between the centers; in which case, the circles are externally apart from each other.

RECENT PUBLICATIONS.

REVIEWS.

*The Philosophy of Mr. B*rtr*nd R*ss*l, with an appendix of leading passages from certain other works.* By P. E. B. JOURDAIN. London, G. Allen & Unwin, 1919. 96 pages. Price 3s. 6d.

This whimsical little book will be enjoyed by logicians who have not lost all their sense of humor, and by humorists who have not lost all their sense of logic. For others the humor may seem rather heavy and labored, and the logic a little more so. There is much quoting of the immortal works of Lewis Carroll,—indeed the Red Queen furnishes the motto for the book, "Even a joke should have some meaning,"—and many mildly amusing anecdotes are given. The following quotation from the chapter on the use of the Identity in Logic will give an idea of the spirit of the book:

"Mr. Austin Chamberlin, according to the *Times* of March 27th, 1909, professed to deduce the conclusion that it is not right that women should have votes from the premisses that 'man is man' and 'woman is woman.' This method requires that one should have made up his mind about the conclusions before discovering the premisses by what, no doubt, Jevons would call an 'inverse or inductive method.' Thus the method is of use only in speeches and in giving good advice. Mr. Austin Chamberlin afterwards rather destroyed one's belief in the truth of his premisses by putting limits to the validity of the principle of identity. In the course of the Debate on the Budget of 1909 he maintained against Mr. Lloyd George, that a joke was a joke except when it was an untruth; Mr. Lloyd George, apparently, being of the plausible opinion that a joke is a joke under all circumstances."

D. N. LEHMER

A short course in college mathematics comprising thirty-six lessons in Algebra, Coördinate Methods, and Plane Trigonometry. By R. E. MORITZ. New York, Macmillan, 1919. 12mo. 9 + 236 pp. Price $2.00.

Extracts from the Preface: "There has been an increasing demand in recent years for shorter courses in mathematics, based on the assumption, which our experience during the Great War has to some extent verified, that the ordinary processes of education may be greatly accelerated. . . .

"This little text is based on the supposition that such condensation and acceleration is possible. It was first prepared and printed for use in the mathematics classes of the Army and Navy Students Training Corps. . . .

"The book contains but thirty-six lessons, of which eighteen are given to the subject of trigonometry and the other eighteen to topics in algebra, to graphs, and to coördinate methods.

Yet it is hoped that no essential principle of elementary trigonometry has been omitted, and that this subject has been treated with a completeness sufficient for the needs both of the engineer and the student of the more advanced branches of mathematics, such as analytical geometry and the calculus."

Contents: Chapter I, Algebra: Factoring; radicals, fractional and negative exponents; imaginary quantities; quadratic equations; applied problems in quadratic equations; some problems in gunnery; review, 1–43. II, Graphic methods: Coördinates and simple graphs; related graphs; straight line graphs; simultaneous straight line graphs; curve plotting; maxima and minima; areas; the straight line and circle; graphic solution of equations; review,44–109. III, Trigonometric functions: The general angle and its measures; the trigonometric or circular functions; reductions to the first quadrant; functions of an acute angle; trigonometric graphs; solution of right triangles; arithmetic solution of oblique triangles; functions of the sum and difference of two angles; inverse trigonometric functions and trigonometric equations; review, 110–178. IV, Logarithms and their use: Exponents and logarithms; logarithmic computation; application of logarithms to mensuration of plane figures; logarithmic and exponential curves, 179–203. V, Logarithmic solution of triangles: Oblique triangles, cases I, II, III, IV; miscellaneous problems involving triangles; review, 204–232. Index, 233–236.

The Mystery of Space. A Study of the Hyperspace Movement in the Light of the Evolution of New Psychic Faculties and An Inquiry into the Genesis and Essential Nature of Space. By R. T. BROWNE. New York, Dutton, 1919. 8vo. 18 + 395 pp. Price $4.00.

Preface, first paragraph: "Mathematics is the biometer of intellectual evolution. Hence, the determination of the *status quo* of the intellect at any time can be accomplished most satisfactorily by applying to it the rigorous measure of the mathematical method. The intellect has but one true divining rod and that is mathematics. By day and by night it points the way unerringly, so long as it leads through materiality; but, falteringly, blindly, fatally, when that way veers into the territory of vitality and spirituality."

Contents: Introduction, explanatory notes, 1–22; Chapter I, The prologue, 23–43; II, Historical sketch of the hyperspace movement, 44–68; III, Essentials of the non-euclidean geometry, 69–91; IV, Dimensionality, 92–117; V, The fourth dimension, 118–160; VI, Consciousness the norm of space determinations, 161–202; VII, The genesis and nature of space, 203–241; VIII The mystery of space, 242–283; IX, Metageometrical near-truths, 284–326; X, Media of new perceptive faculties, 327–358; Bibliography, 359–366; Index, 367–395.

NOTES.

In *General Mathematics,* [for first year in a high school] (Boston, Ginn, 1919. 12mo. 16 + 488 pp. $1.48), by R. SCHORLING and W. D. REEVE, members of the Association, "the material purposes to present such simple and significant principles of algebra, geometry, trigonometry, practical drawing, and statistics, along with a few elementary notions of other mathematical subjects, the whole involving numerous and rigorous applications of arithmetic as the average man (more accurately the modal man) is likely to remember and to use."

Macmillan, London, has published volume III of Sir Thomas Muir's *The Theory of Determinants in the Historical Order of Development,* covering the period 1861–1880.

Giornale di Matematica Finanziaria is the title of a quarterly started in 1919 under the direction of Doctors F. Insolera and S. Ortu-Carboni (Publisher, 73 Corso Vittorio Emmanuele, Turin, Italy; price 16 lire a year). The editors plan that the periodical shall contain not only purely scientific studies with special reference to mathematics of finance (credit, insurance, statistics, etc.), but also reviews of books and periodicals, as well as of laws, decrees, and regulations.

ARTICLES IN CURRENT PERIODICALS.

ACTA MATHEMATICA, volume 42, no. 3, 1919: "Gaston Darboux (1842–1917)" by D. Hilbert, 269–273 ["Traduction du discours prononcé le 12 mai 1917 à la séance publique annuelle de l'Académie des sciences de Goettingue (*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1917, p. 71)"]; "Darboux's Anteil an der Geometrie" by L. P. Eisenhart, 275–284 ["Uebersetzung der am 6. September 1917 vor der vereinigten Sitzung der 'American Mathematical Society' und der 'Mathematical Association of America' in Cleveland gehaltenen Vorlesung (*Bulletin of the American Mathematical Society*, volume 24, 1918, p. 227)"].

AMERICAN JOURNAL OF MATHEMATICS, volume 41, no. 4, October, 1919: "The ten nodes of the rational sextic and of the Cayley symmetroid" by A. B. Coble, 243–265; "Functions of matrices" by H. B. Phillips, 266–278; "On the Lüroth quartic curve" by F. Morley, 279–282; "On the order of a restricted system of equations" by F. F. Decker, 283–298; "On the Lie-Riemann-Helmholz-Hilbert problem of the foundations of geometry" by R. L. Moore, 299–319.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 26, no. 3, December, 1919: "Note on convergence tests for series and on Stieltjes integration by parts" by R. D. Carmichael, 97–102; "Note on a physical interpretation of Stieltjes integrals" by R. D. Carmichael, 102–105; "A derivation of the equation of the normal probability curve" by W. D. Cairns, 105–108; "Bôcher's boundary problems for differential equations" [review of M. Bôcher's *Leçons sur les Méthodes de Sturm dans la Théorie des Equations Différentielles Linéaires et leurs Développements Modernes*, . . . edited by G. Julia (Paris, 1917)] by R. G. D. Richardson, 108–124; "Dickson's *History of the Theory of Numbers*" [review of Vol. I, Washington, 1919] by D. N. Lehmer, 125–132; "The calculus of probability" [review of G. Castelnuovo's *Calcolo delle Probabilità* (Milano-Roma-Napoli, 1919)] by R. D. Carmichael, 132–135; "Corrections," 136–137; "Notes," 136–141; "New publications," 142–144.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY, Volume 9, no. 2, March, 1919: "The late Sir Gooroodas Banerjee" by S. K. Banerji, i–v and portrait frontispiece ["By the death of Sir Gooroodas Banerjee, which melancholy event took place on the 2d of December, 1918, the Calcutta Mathematical Society has lost not only its senior Vice-President but one who took the deepest interest in all its proceedings, one whose great learning, splendid endowments of head and heart, deep piety, purity of character and loftiness of aims and principles were always object-lessons to its members. Manifold as were the activities of Sir Gooroodas Banerjee, his life is one of considerable interest to the mathematician. Sir Gooroodas began his life as a teacher of mathematics and ended it as an author of mathematical text-books. Whether as a lawyer, a judge, or an educationist, Sir Gooroodas owed, in no inconsiderable degree, his sound judgment, his accurate logic and his strong common sense to his high attainments as a mathematician.

"Sir Gooroodas was born on the 27th of January, 1844, in Narikeldanga, in the suburbs of Calcutta. . . . Many years ago he wrote a book on the Elements of Arithmetic and Algebra. For a long time he could hardly find time and leisure to pursue this favourite subject of his and it was only after his retirement from the Bench that he could renew his mathematical activities. . . . [He] brought out in 1906 his *Elementary Geometry* adapted to modern methods. This book is a small volume of 142 pages and comprises within its limits the substance of the first six books of Euclid and almost all the elementary propositions of Solid Geometry. The most attractive feature of the book is that a beginner can learn with ease and within a short time all the important elementary truths of Geometry. This book has already passed through several editions and can be seen in the hands of almost all our school-going boys. . . . Quite recently Sir Gooroodas formed an elaborate plan of bringing out a series of mathematical text-books in our own vernacular Bengali under the name *Saral Ganit* and published three parts dealing respectively with Arithmetic, Algebra, and Geometry based on modern methods. All three volumes are characterized by the same remarkable simplicity and conciseness which form the distinguishing features of his English treatise on Elementary Geometry "]; "On the figures of equilibrium of two rotating masses of fluid for the exponential potential e^{-kr}/r," Part 1 by A. Datta, 59–70; "Fourier's series and its influence on some of the developments of mathematical analysis" by A. C. Bose, 71–84 ["In the *Bulletin of the Calcutta Mathematical Society* (vol. 7, pp. 33–48), an account has been given of the Life and Work of Fourier. The present paper endeavours to state briefly how some of the concepts of modern mathematics of a far-reaching character, arose out of Fourier's Analysis, justifying P. E. B. Jourdain's remark that 'if it is safe to trace back to any single man the origin of those conceptions with which pure mathematical analysis has been chiefly occupied during the nineteenth century and up to the present time, we must, I think, trace it back to Jean Baptiste Joseph Fourier

(1768–1830).' . . . He died full of honours and to the infinite regret of his colleagues who had come to appreciate his eminence as a physicist and a mathematician. As Arago said 'his life had been so varied, so laborious, so gloriously interlaced with the greatest events of the most memorable epochs of French History, that the scientific discoveries of the illustrious Secretary [of the Academy of Sciences] had nothing to dread from the incompetence of the panegyrist.'

"We bow then to the greatness of Jean Baptiste Joseph Fourier and we greet him, in the 20th century, with the eloquent words of Sir William Rowan Hamilton, another great Mathematician whom Fourier charmed and inspired in writing of fluctuating functions:

"Fourier with solemn and profound delight,
Joy born of awe, but kindling momently
To an intense and thrilling ecstasy,
I gaze upon thy glory and grow bright:
As if irradiate with beholden light;
As if the immortal that remains of thee
Attune me to thy spirit's harmony,
Breathing serene resolve and tranquil might,
Revealed appear thy silent thoughts of youth,
As if to consciousness, and all that view
Prophetic, of the heritage of truth
To thy majestic years of manhood due:
Darkness and error fleeing far away,
And the pure mind enthroned in perfect day."];

"On the numerical calculation of the roots of the equations $P_n^m(\mu) = 0$ and

$$\frac{d}{d\mu} P_n^m(\mu) = 0$$

regarded as equations in n" by B. Pal, 85–95; "On some new theorems in the geometry of masses" by S. Dhar, 96–107; "On the electric resistance of a conducting spheroid with given electrodes" by S. Kar, 109–114—Volume 10, no. 1, June, 1919: "On surface waves and tidal waves near a promontory" by S. Banerji, 1–10; "On the potentials of uniform and heterogeneous elliptic cylinders at an external point" by N. Sen, 11–27; "Notes on inversion" by T. Bhattacharyya, 29–34; "On the use of Ritz's method for finding the vibration-frequencies of heterogeneous strings and membranes" by N. K. Majumdar, 35–42; "On the steady motion of a viscous fluid due to the rotation of two rigid bodies about arbitrary axes" by B. Dutt, 43–61; "Obituary Notices," 63 [1. The late Mr. Chandrashekkar Sircar; 2. The late Principal Ramendrasundar Trivedi]—No. 2, September: "New methods in the geometry of a plane arc. II—Cyclic points and normals" by S. Mukhopadhyaya, 65–72; "Origin of the Indian cyclic method for the solution of $Nx^2 + 1 = y^2$" by P. C. Sen-Gupta, 73–80 [First and last paragraphs: "The object of the present paper is to discuss the probable origin of the 'Cyclic Method' (*Chakravala*) for the solution of $Nx^2 + 1 = y^2$ in rational integers as given in Bhaskara's *Vijaganita*. Two hypotheses have been advanced as regards its origin: first that the method has an ultimate Greek source and secondly that it is purely Indian. I shall first discuss the former view and shall next show that it is untenable in the light of the reasons which, I trust, are put forth herein for the first time. . . . It is thus seen that to arrive at the Indian cyclic rule it is not at all necessary to determine an approximation to $\sqrt{N}$ either by the Archimedian method or by any other method. It is further evident that the rules are immediate deductions from the lemma of Brahmagupta and the sole credit of finding a method for the solution of $Nx^2 + 1 = y^2$ belongs to him"]; "On the motion of an ellipsoid of revolution in a viscous fluid in the light of Prof. Oseen's objection to Stokes's treatment of the case of the sphere" by B. Pal, 81–94 ["The motion of a sphere in a viscous fluid has been investigated by various writers, including Stokes, Profs. Whitehead, Oseen, Lamb, Burgess, the results obtained being more or less satisfactory according to the degree of approximation to which the differential equations are satisfied.

"In the present paper, I propose (1) to obtain the solution of the problem of the motion of translation of an ellipsoid of revolution of small ellipticity in a viscous fluid, the method adopted being similar to that of Prof. Lamb for treating the corresponding problem in the case of the sphere, and (2) to show how the results obtained by me although different in some respects from those given by Oberbeck, the only important writer who investigated the ellipsoidal problem before me, are free from any objection similar to that pointed out by Prof. Oseen in Stokes's treatment of the spherical problem."]; "On a class of ellipsoidal harmonics and a method of

solving the wave equation in ellipsoidal coordinates" by S. Banerji, 95–104; "Some cases of tidal oscillations in canals of variable section" by S. Dasgupta, 105–116; "The stress-equations of equilibrium" by S. Basu, 117–121; Review by S. M. of Miller, Blichfeldt, and Dickson's *Theory and Applications of Finite Groups* (New York, 1916), 123–124.

COLORADO COLLEGE PUBLICATIONS, science series, volume 12, no. 3, 1907: "On the transformation of algebraic equations by Erland Samuel Bring (1786)," translated and annotated by F. Cajori, 63–90—No. 7, 1910; "A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity" by F. Cajori, 171–287—No. 15, November, 1919; "On non-ruled octic surfaces whose plane sections are elliptic" by C. H. Sisam, 639–655.

CREIGHTON-COURIER, Creighton University, Omaha, Nebraska, volume 8, No. 12, February 7, 1920: "Einstein's theory of gravitation" by W. F. Rigge, 2–3.

EDUCATIONAL REVIEW, volume 59, January, 1920: "Mathematics as a study" by G. S. Painter, 19–40. [First paragraph: "It is a certainty that educational wisdom will not die with this generation. Never in our history perhaps have such perplexity and indecision prevailed as now relative to the ends and methods of education. Nothing is any longer simple or fixt; everything is in a state of flux. The parvenu educational iconoclast is abroad in the land, and with opprobrious assurance hesitates not to lay unholy hands upon our most sacred educational traditions and institutions. No science or subject of investigation is any longer of value in and of itself, but only in so far as it contributes to something else. The implicit motive back of all this attitude is one of vulgar utility which is regarded as the sole and only reason for the existence of anything. We are, then, somewhat in a state of educational anarchy in which, with characteristic unwisdom, the blind are leading the blind with the conventional result of all getting into the ditch."]

L'ENSEIGNEMENT MATHÉMATIQUE, volume 20, no. 5, October, 1919: "Sur une transformation élémentaire et sur quelques intégrales définies et indéfinies" by C. Cailler, 317–337; "Sur l'intégrale $n!\int_0^h \frac{h^n e^{-(hx/1-h)}}{1-h}\,dh$" by F. Vaney and M. Paschoud, 338–346; "Extension de la notion de Jacobien" by M. Stuyvaert, 347–354; "Sur la représentation proportionnelle en matière électorale" by G. Pólya, 355–379 [First paragraph: "Dans plusieurs périodiques non mathématiques,[1] j'ai essayé de mettre en contact l'analyse mathématique avec l'énorme diversité des opinions émises sur la question de la représentation proportionnelle en matière électorale. La partie la plus intéressante de la recherche est, me semble-t-il: trouver, dans une littérature de controverse qui s'éloigne beaucoup de l'exposition et des sujets mathématiques habituels, des principes tangibles, des faits susceptibles d'une explication exacte et les 'mettre en équation.' Dans les travaux cités j'ai énoncé plusieurs résultats mathématiques. Je les ai vérifiés expérimentalement par des exemples, j'ai tâché de les rapprocher du bon sens sans l'aide des formules, mais j'ai dû omettre les démonstrations. Dans les lignes suivantes, je donnerai l'analyse exacte, une analyse très élémentaire d'ailleurs, mais qui ne sera peut-être pas dépourvue d'un certain intérêt pour quelques lecteurs"]; "Mélanges et correspondance" ["Au sujet de la publication de mon article sur 'Les origines d'un problème inédit de E. Torricelli' (*L'Enseignement Mathématique*, vol. 20, pp. 245–268), je dois signaler que M. Michele Cipolla, professeur à l'Université de Catane, vient de faire paraître une important étude sur le même problème: Michele Cipolla, "I triangoli di Fermat e un problema di Torricelli," *Atti dell'Accademia Gioenia di scienze naturali in Catania*, serie 5, vol. 10, memoria XI . . ." Emile Turrière.]; "Chronique;" "Notes et documents;" Review by C. A. Laisant of G. Bouligand's *Cours de géométrie analytique* (Paris, 1919), 390–391; Review by A. Buhl of P. Boutroux's *Les principes de l'analyse mathématiques*, tome 2 (Paris, 1919), 391–392; Review by E. Turrière of F. C. Clapier's *Sur les surfaces minima ou élassoides* (Paris, 1919), 392–393; Review by A. Buhl of G. H. Halphen's *Œuvres*, and R. de Montessus de Ballore's *Introduction à la théorie des courbes gauches algébriques* (Paris, 1918), 393–397; "Bulletin Bibliographique."

JOURNAL OF EDUCATIONAL RESEARCH, University of Illinois, volume 1, no. 1, January, 1920: "Hurdles, a series of calibrated objective tests in first year algebra" by M. A. Dalman, 47–62.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 11, no. 5, October, 1919: "A problem of diophantine approximation" by G. H. Hardy, 162–166; "A general theorem

[1] *Schweiz. Zentralblatt für Staats- und Gemeindeverwaltung*, 1919, no. 1; *Journal de statistique suisse*, 1918, no. 4; *Wissen und Leben*, January and February, 1919. *Zeitschrift fur die gesamte Staatswissenschaft* (sous presse).

relating to the cartesian oval" by A. C. L. Wilkinson, 167–172; "Multiplication of infinite integrals" by K. B. Madhava, 173–180; "A proof of Bertrand's postulate" by S. Ramanujan, 181–182; "Astronomical notes" by T. P. Bhaskara Sastri, 183–184; Problems and solutions, 185–200; "Mathematics from periodicals" by V. B. Naik, i–v.

MATHEMATICAL GAZETTE, volume 9, October, 1919: "The graphical treatment of differential equations" by S. Brodetsky, 377–382 (to be continued) [*First two paragraphs:* "The subject of this paper is one that presented itself in a piece of work of a practical nature. The development of aeroplane flight naturally suggested the investigation of the motion of a body in a resisting medium. In general this problem was too difficult for solution. One therefore looked for legitimate means of simplifying it. One simple type of motion allied somewhat to areoplane motion was that of a plane lamina moving in air under the earth's attraction. Even here difficulties arose in the solution of the resulting equations of motion; yet further simplification was not permissible, for care had to be taken that the simplified problem did bear some relation to the actual problem presented by nature. It was ultimately found that the simplest possible form of the problem gave rise to the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} - (x^2 + y^2)^{1/2},$$

or, putting $(x^2 + y^2)^{1/2} = r$,

$$\frac{dr}{dx} = -y.$$

The disconcerting feature about this equation was the fact that it was found to be quite insoluble by any of the 'standard forms' dealt with in books on differential equations. Several mathematicians tried to discover a 'transformation' or an 'integrating factor,' but without success. Yet a solution had to be found somehow.

"It then occurred to the writer that where analysis had failed, geometry might succeed. The solution of a differential equation of the first order is of course represented geometrically by a family of curves. The ordinary treatment of differential equations consists in seeking for an analytical representation of these curves. Since, then, the analytical formula was apparently unobtainable, might not the curves themselves be 'graphable'? The result was a complete success"]; "Coördinate geometry in schools" by W. J. Dobbs, 383–388 (to be continued); Reviews of F. S. Woods and F. H. Bailey's *Analytic Geometry and Calculus* (Boston, 1918), H. B. Phillips's *Differential and Integral Calculus* (New York, 1916–1917), H. Hancock's *Theory of Maxima and Minima* (Boston, 1917), H. Bateman's *Differential Equations* (London, 1918), and *Annuaire pour l'an* 1919, (Bureau des Longitudes, Paris, 1919) 389–391 [*Last paragraph of last mentioned review:* "The essays which are always a notable feature in the Annuaire, are this year by MM. Appell and Hamy. The former finds a congenial topic in 'The figures of relative equilibrium of a rotating homogeneous liquid' (60 pp.). After an historical sketch of the theory up to the work of the late MM. Liapounoff and Poincaré, he deals with the problem of stability and the phenomena at the point of bifurcation. The value of the paper is greatly enhanced by a full bibliography. M. Hamy treats of the inference of the real diameters of minor planets and satellites from a study of the interference fringes (27 pp.). He believes that, with the more powerful instrumental opportunities at our disposal in the 100 in. reflector at Mount Wilson, it will be quite possible to determine the angular-diameters of 1st magnitude stars. The excellent table of contents runs to 69 pp.].

MESSENGER OF MATHEMATICS, volume 48, no. 11, March, 1919: "The dissection of rectilineal figures "(conclusion) by W. H. Macaulay, 161–165; "On a Diophantine problem" (second paper) by H. Holden, 166–176—No. 12, April: "On a Diophantine problem" (conclusion) by H. Holden, 177–179; "Sur quelques intégrales definies" by S. P. Shensen, 179–184; "On n-poled cassinoids" by H. Hilton, 184–192 [The polar equation of the curve is $r^{2n} - 2r^n c^n \cos n\theta + c^{2n} = a^{2n}$].

NATURE, volume 104, November 20, 1919: "Percussion figures in isotropic solids" by J. W. French, 312–314—November 27: "Gravitation and light" by O. J. Lodge, 334 [*The note:* "It may or may not have been noticed that the refractivity $(\mu - 1)$ at any point, required to produce the Einstein deflection, is the squared ratio of the velocity of free fall from infinity to the velocity of light."]; "A new astronomical model" by A. L. Cortie, 343–344. [*First paragraph:* "The illustrious scholar Gerbert (A.D. 940–1003), afterwards Pope under the name of Sylvester II., was apparently the first of the schoolmen who illustrated his theoretical lessons on astronomy by the use of globes, which he constructed with his own hands. About the year A.D. 1700 George

Graham invented a machine to show the movements of the earth and planets about the sun, a copy of which was made by Charles Boyle, the Earl of Orrery. Hence the name of an apparatus very useful for illustrating lessons in astronomy, although Sir John Herschel did call orreries 'very childish toys.' But surely the difficulty in teaching astronomy is to make the young pupil think in three dimensions. What are we going to do when the relativists would have us imagine phenomena in four dimensions?" *Last paragraph:* "Dr. Wilson is to be heartily congratulated on having produced such a valuable, workable astronomical model. So many science masters—excellent men!—have desired to acquire it that he has felt justified in putting it upon the market and getting it made in quantities. The price is 22*l.* net, carriage paid to any part of the United Kingdom. All communications regarding the model should be addressed to Dr. Wilson himself at 43 Fellows Road, London, N.W. 3."].—December 4: "Gravitation and light" by O. J. Lodge, 354; "The displacement of light rays passing near the sun" by A. Anderson, 354; "Einstein's relativity theory of gravitation" by E. Cunningham, 354–356; [Note], 360 ["The *Times* of November 28 contains an article from Prof. Einstein on his generalized principle of relativity. Prof. Einstein remarks at the beginning of the article: "After the lamentable breach in the former international relations existing among men of science, it is with joy and gratefulness that I accept this opportunity of communication with English astronomers and physicists. It was in accordance with the high and proud tradition of English science that English scientific men should have given their time and labour, and that English institutions should have provided the material means, to test a theory that had been completed and published in the country of their enemies in the midst of war.' After a brief account of the general nature of the theory, which does not add anything to what has been summarised by Prof. Eddington in his report to the Physical Society, Prof. Einstein concludes: 'The great attraction of the theory is its logical consistency. If any deduction from it should prove untenable, it must be given up. A modification of it seems impossible without destruction of the whole. No one must think that Newton's great creation can be overthrown in any real sense by this or any other theory. His clear and wide ideas will for ever retain their significance as the foundation on which our modern conceptions of physics have been built. . . . By an application of the theory of relativity to the taste of readers, to-day in Germany I am called a German man of science, and in England I am represented as a Swiss Jew. If I come to be regarded as a *bête noire*, the descriptions will be reversed.' Prof. Eddington, in the *Contemporary Review*, quotes from Newton's *Opticks*:—'Query 1. Do not bodies act upon light at a distance, and by their action bend its rays?'"]; [Einstein theory at the anniversary meeting of the Royal Society], 361–362.—December 11: "Gravitation and light" by O. J. Lodge, 372 ["Jupiter ought just to show the Einstein deflection, for if it pass between two stars a couple of diameters of the planet apart, their temporary relative displacement will be a 'third' of arc, the sixtieth of a second; and this could be measured with a heliometer"]; "The deflection of light during a solar eclipse" by A. S. Eddington and A. C. D. Crommelin, 372–373; "Einstein's relativity theory of gravitation. II—The nature of the theory" by E. Cunningham, 374–376—December 18: Review by S. Brodetsky of Maria M. Roberts and Julia T. Colpitts's *Analytic Geometry* (New York, 1918), P. Goyen's *Elementary Mensuration, constructive plane geometry, and numerical Trigonometry* (London, 1919) and J. B. Shaw's *Lectures on the Philosophy of Mathematics* (Chicago, 1918), 390–391; "Deflection of light during a solar eclipse" by W. H. Dines, L. F. Richardson, and A. Anderson, 393–394. "Einstein's relativity theory of gravitation. III.—The crucial phenomena" by E. Cunningham, 394–395.—January 22, 1920: "Gravitation and light" by J. Larmor, 530; "The Einstein theory and spectral displacement," two notes by H. F. Moulton and A. C. D. Crommelin, 532—January 29; "The works of Toricelli" by J. L. E. D., 557–558; "The deflection of light during a solar eclipse" by A. Anderson, 563; "Displacement of spectral lines" by R. W. Lawson, 565 ["In view of the discussion in *Nature* and elsewhere on this subject, the following extract from a recent letter of Prof. Einstein may be of interest: 'Zwei junge Physiker in Bonn haben nun die Rot-Verschiebung der Spektral-Linien bei der Sonne so gut wie sicher nachgewiesen und die Gründe des bisherigen Misslingens aufgeklärt.' I have heard no details, but doubtless an account of this work will be available before long"]—February 5: "The predicted shift of the Fraunhofer lines," by J. Rice and A. S. Eddington, 598–599; "The straight path" by A. A. Robb, 599; "Mathematics in the United States" by G. B. M[athews], 601—February 12: "Euclid, Newton, and Einstein" by W. G., 627–630; "The theory of relativity" by A. C. D. Crommelin, 631–632.

Philosophical Magazine, volume 38, November, 1919: "The Bessel-Clifford function and its applications" by G. Greenhill, 501–528.

REVUE DE L'ENSEIGNEMENT DES SCIENCES, volume 12, November–December, 1918 (published February, 1919): "Le cas singulier des fonctions implicites et les enveloppes dans le plan" by G. Bouligand, 225–237; "Sur le dessin géométrique" by C. Bioche, 238–240; "Sur le deuxième principe fondamental de la méthode infinitésimale" by G. Lapointe, 240–243; "Notes de géométrie analytique" by C. Michel, 243–249; "Des différents systèmes de numération. Propriétés des nombres dans ces divers systèmes" (suite) by R. Massart, 249–259 [Paragraphs 47–59; the article was started in the first number of the volume]; "Correspondance," 259; "Examens et concours de 1918," 260–266—Volume 13, January–February, 1919 (published, May): "Sur la loi exponentielle d'erreur, les séries d'observations et la moyenne arithmétique dans les sciences physiques" by L. Genillon, 1–11; "Sur les ovals de Descartes" by L. Genillon, 11–18; "Une propriété de la parabole" by J. Lemaire, 23–26; "Sur une surface réglée du quatrième ordre" by J. Lemaire, 27–30; "Sur les systèmes de numération et le calcul des polynomes" by J. Juhel-Rénoy, 30–33; "Des différents systèmes de numération. Propriétés des nombres dans ces divers systèmes" (suite et fin) by R. Massart, 33–43; "Examens et concours de 1918," 43–64—March–April: "Les trois dispositions de deux trièdres supplémentaires" by G. Fontené, 65–68; "Sur un procédé de calcul et son application à la topographie" by T. Leconte 68–88; "Sur les rayons de courbure et les normales polaires" by C. Bioche, 88–90; "La rotation et le quadrilatère inscriptible" by Amsler, 90–92; "Problèmes de mathématiques et de physiques donnés au baccalauréat, 1918," 93–95, 109–118.

REVUE DE PHILOSOPHIE, volume 19, no. 5, September–October, 1919: "L'infini mathématique" (second article) by P. M. Périer, 516–533.

SCIENTIA, volume 26, December, 1919: "Il posto di Leonardo nella storia delle scienze" by A. Favaro, 437–448, supplément 137–149; Review by G. Loria of M. Bôcher's *Leçons sur les méthodes de Sturm dans la théorie des équations différentielles linéaires* (Paris, 1917).

SCIENTIFIC MONTHLY, volume 9, no. 6, December, 1919: "Our universe of stars" by E. Doolittle, 506–513; "Individuality in research" by R. D. Carmichael, 514–525 [*First paragraph:* "The men of no other country have exhibited so great a measure of individuality in research as those of England. In the highest degree they have manifested the self-reliant strength of natural genius. Among them scientific endeavor has not been organized; and for the most part investigations have been carried out by single workers in isolation. British science has always refrained from congregating in distinct schools and institutions; it has never been localized in definite centers. No compact body of pupils there develops the work and ideas of any master. Thinkers proceed singly and individually with their self-appointed tasks laboring in close touch with nature and according to their particular inclinations."]—Volume 10, no. 3, March, 1920: "Space, time and gravitation" by E. B. Wilson, 217–235 [Read before the Royce Club, Boston, January 18, 1920]; "Finis coronat opus" by F. V. Morley, 306–308 [First paragraph: "The evening vigil may be approached in many ways. Perhaps the pleasantest of all is to fortify the soul with plenty of tobacco, a canister of crackers and something potable to fall back upon, and a volume of the inimitable Robert Burton. But failing the 'merry companionship' of the younger Democritus, we should be prone to choose the scientific remains of a poor, neglected, and forgotten scholar whose name was also Robert [Recorde], but who was earlier in his unhappy life at Oxford by some fifty years"].

THEOSOPHICAL PATH, Point Loma, Cal., volume 18, no. 2, February, 1920: "Neglected foundations of geometry" by F. J. Dick, 160–171 [Last paragraph: "To go on telling people that it is impossible to 'square the circle' by the plane geometry of line and circle is surely as absurd as the evaluation of π (actually accomplished) to 700, or even to 30 decimal places, seeing that no radius, whether it be the 'astronomical unit' or the radius of an engine-cylinder, is ever known with greater accuracy than that defined by the fifth or sixth significant figure, at the most."]

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 20, no. 4, October, 1919 (published January, 1920): "Line-geometric representations for functions of a complex variable" by E. J. Wilczynski, 271–298; "On the combination of non-loxodromic substitutions" by E. B. Van Vleck, 299–312; "On a general class of integrals of the form $\int_0^\infty \phi(t)g(x+t)dt$" by R. D. Carmichael, 313–322; "Transformations of surfaces applicable to a quadric" by L. P. Eisenhart, 323–338; "Note on two three-dimensional manifolds with the same group" by J. W. Alexander, 339–342; "A general system of linear equations" by Anna J. Pell, 343–355; Errata, 356; Index by authors, index by subject matter, volumes 11–20, 31 pp. [There are 137 authors. Ten papers are listed under the heading, "Logical analysis of mathematical disciplines;" thirty-

eight under the heading "Algebra"; thirteen under "Theory of numbers"; one hundred and twenty under "Analysis"; twenty-nine under "Groups"; seventy-six under "Geometry"; twelve under "Applied mathematics."]—Volume 21, no. 1, January, 1920: "The strain of a gravitating sphere of variable density and elasticity" by L. M. Hoskins, 1–43; "The geometry of hermitian forms" by J. L. Coolidge, 44–51; "Certain types of involutorial space transformations" by F. R. Sharpe and V. Snyder, 52–78.

AMERICAN DOCTORAL DISSERTATIONS.

E. F. SIMONDS, "Invariants of differential configurations in the plane," *Transactions of the American Mathematical Society*, 1918, volume 19, pp. 222–250 (Columbia, 1917).

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB TOPICS.

Although much has already been printed concerning the abacus and its uses,[1] we believe that our readers will find the following article by Professor Leavens decidedly interesting and helpful since it gives an independent discussion based upon the personal impressions and first-hand information of a westerner who has come into contact with the present-day use of the abacus in the far east.

17. THE CHINESE SUAN P'AN.

By DICKSON H. LEAVENS, College of Yale in China.

The Chinese suan p'an[2] or abacus is familiar to many from an occasional sight of it on a laundryman's table, but it is perhaps usually regarded either as a device full of the mystery of the East and beyond the grasp of the Occidental, or as an instrument fit only for the ignorant "Celestial" and beneath the notice of one who has studied arithmetic.

A little investigation, however, will show one that it is not only perfectly

[1] Two of the best discussions in English are probably C. G. Knott's article, cited below, and Leslie's *Philosophy of Arithmetic* (Edinburgh, 1820), pp. 15–100. Leslie gives, in great detail, examples of the representation of numbers in different scales of notation and of operations by means of them. From his discussion one can readily see how certain theorems on divisibility of numbers and even the summation of special infinite descending series may be inferred from the use of the abacus.

Some excellent illustrations and references to the literature of the abacus can be found in Smith and Mikami's *History of Japanese Mathematics* cited below. Other readily accessible sources of information are the descriptions given in current histories of mathematics and articles in encyclopedias under the titles "Abacus" and "Calculation."

[2] In this MONTHLY (1919, 256), it is noted that this word, with various spellings, appears in the *New English Dictionary*, but that "the Chinese word Soroban . . . is not given." Soroban, however, is not Chinese, but Japanese, being the Japanese pronunciation of the same characters, which are used in the written language of both countries.

The same note (following the dictionary) translates suan p'an wrongly as reckoning *board*. The Chinese character for board is romanized pan, hence the error; but it is quite a different character from the one here used, which means a *plate* or *tray;* the pronunciation is also different, p in the most used system of romanization representing practically our b, while p' is similar to our p.

simple, but that it is also not to be despised as an aid in arithmetical work when an adding machine is not available.

The instrument consists simply of a rectangular frame, usually with a back so that it forms a kind of tray, containing a number of rods, about 1/8 inch in diameter, made of wood, bone, or metal, set parallel to each other and to the shorter dimension of the frame. On each rod are strung seven wooden beads, the upper two separated from the lower five by a "bridge" which runs the whole length of the frame. In the Chinese suan p'an, the beads are roughly in the shape of a 60° equatorial zone section of a sphere; in the Japanese soroban, they are in the form of a double cone, with a rather sharp angle where the bases join, and it is claimed that this sharp edge makes for quicker manipulation than the rounded Chinese form.

The most usual size of the instrument is about seven by fourteen inches, and contains thirteen rods or columns, with beads about 7/8 inch in diameter. They may also be obtained with more columns, or with smaller beads, down to about 3/8 inch in diameter; but these are too small for convenience, and the standard size, or a little smaller, is the most practicable. The cost is from fifty cents to a few dollars, depending on the size, and on the quality of the wood.

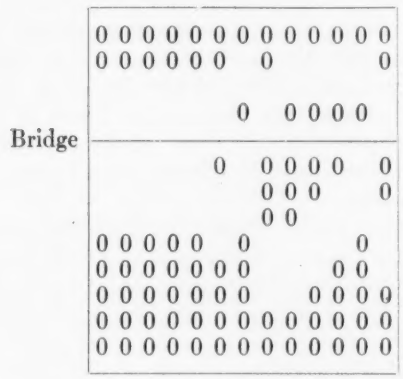
Each rod represents one column, arranged in decreasing powers of 10 from left to right, as in our system of notation. No decimal point is marked on the ordinary instrument, the user locating it mentally for the problem he is working on. Banks sometimes have the columns labelled, to facilitate computations in the exchange of money. The westerner will find it convenient to tie strings between the columns, separating them into groups of threes; coloring the beads differently in groups of three columns, as is done with the keys of adding machines, might also be a help.

In each column, each of the lower five beads represents 1 unit, and each of the upper two represents 5 units. Properly, only four beads below the bridge, and one above are needed to set up 9 in each column; the Japanese soroban does omit one of the upper beads, thus allowing only 10 to be set up. The Chinese form permits the setting up of 15 in each column; this is never done in addition, as 10 is at once carried to the next column on the left, but in long division it is a convenience to have the extra capacity in setting up the steps of the problem.

It is in this use of one upper bead to represent 5 units that the suan p'an is far superior as a practical device to the abacus of ten beads, often sold as a children's toy or for use in primary arithmetic. On the suan p'an, the eye can take in at a glance the number in a group of beads, never more than four; while on the ten bead form, it is often necessary to stop and count. The arrangement in vertical columns is also a preparation for the decimal notation, much better than the horizontal rows of the American form. Still another advantage is that the beads slide easily on the wooden rods, while in the American form, the thin wires are easily bent, and the beads do not slide so smoothly.

Before beginning a problem, the lower beads are slid down by tilting the instrument, and the upper ones are pushed up with a sweep of the hand, leaving

none adjacent to the bridge. Numbers are set up by moving the beads towards the bridge. The figure below represents the number 15387652 set up, the decimal point to be provided according to the conditions of the problem, and there still being five blank columns to the left.



The addition of 1 or 2 to this number is done simply by moving up 1 or 2 beads in the unit column. If it is desired to add 3, moving up 3 beads would make 5 beads below the bridge, and we do not wish to use the 5th bead. But adding 3 is the same as adding 5 and subtracting 2, so we bring down one of the upper beads to the bridge, and push down, or as we might say, "erase" the two lower ones. Similarly to add 4, we add 5 and erase 1, etc. To add 8 to 2 (or to any number greater than 2) we add 1 in the next column to the left, and erase 2, etc. Note that we do not actually get 5 beads below, and then change them for 1 above, etc.; we foresee that carrying is necessary and do it as we add. It follows that the same digit may be added in different ways, according to what digit it is to be added to. For instance to add 7 to 0, 1, or 2, we add 5 and 2; to add 7 to 3, 4, 8, or 9, we subtract 3, and add 1 (*i.e.* 10) in the next column to the left; to add 7 to 5, 6 or 7, we add 2, subtract 5, and add 1 (*i.e.* 10) in the next column to the left. Chinese books of instruction give tables of all possible combinations of two digits, showing just when to add the original number, and when to add 5 or 10, and subtract the complementary part. For a child learning to use the instrument, the learning of these tables is perhaps a necessity; but an adult need never see such tables, and with a little practice will soon learn the combinations, so that the fingers make them automatically with little help from the brain.

The addition of numbers of several digits is not done vertically by columns, as in our system, but horizontally by the complete numbers. The first number is set up, then the digits of the second are added, one by one, working from left to right. Then the third number is added, and so on. The working from left

to right is a distinct advantage, as it gives the digits in the order in which they are written or spoken. This enables one person to use the instrument while another calls off the numbers to him, if necessary; although it is of course more often used by one man alone.

For a fuller description of the method of addition, and also for subtraction, multiplication, division, and extraction of roots, the reader is referred to the full and clear explanations given by Knott,[1] and by Smith and Mikami.[2]

Since the principle of the instrument is perfectly simple, it requires only a little practice to develop fair accuracy and speed in the use of it, at least for addition. It is much simpler to learn than typewriting, but as in that, the best way is by regular daily practice. Fifteen minutes daily will in a short time familiarize one with it, so that the carrying becomes practically automatic. This is one of the features of the instrument that recommends it strongly to the writer. There is none of the mental tension of remembering partial totals. If interrupted, you can stop at any point (provided only you keep track of where you leave off) and take it up again. There is of course the objection that if you make a mistake, you must go back to the beginning, whereas in pencil addition, you have some column totals already written down. But even there, you may have no record of the amounts carried, and so have to start afresh. In any addition, the best check is, probably, to add again. The writer has found the abacus very useful in accounting work, which occupies a good deal of his time. As the columns have to have a pencil total anyway, he first adds up with pencil, and then again by abacus. This gives a very good check, as there is almost no chance of making the same mistake by the two different methods.

Another very useful place for the suan p'an is in adding numbers that are not written in columns. The westerner is trained from childhood to add in columns, and it may not be possible for him to become proficient enough on the abacus to save much time there, although as remarked above the work is made easier. But cross addition of large numbers, or the addition of items picked out here and there in an account book, or of numbers on separate slips, as a pile of checks from the bank, is very difficult for the ordinary person, unless he copies them down in a column. It is here that the abacus saves much time, for the left hand can run along from number to number, or turn over the pile of slips, while the right hand adds. The addition should of course be repeated as a check, preferably taking the numbers in the opposite order. There are cases of this kind where a written list is needed, but when only the total is required the abacus is much quicker.

The writer has used the instrument chiefly for addition, and is doubtful whether the abacus methods of multiplication and division have much advantage

[1] C. G. Knott, "The abacus in its historic and scientific aspects," *Transactions of the Asiatic Society of Japan*, Yokohama, vol. xiv, 1886. This is reprinted in an abridged form, under the title "The calculating machine of the East: the abacus," in Horsburgh, *Modern Instruments and Methods of Calculation, A Handbook of the Napier Tercentenary*, London, Bell, [1911], pp. 136–154.

[2] D. E. Smith and Y. Mikami, *A History of Japanese Mathematics*, Chicago, Open Court, 1914, Chapter III.

over ours, for one who has not used it from childhood. However, the special case of multiplication where a large number of multipliers are to be used on the same multiplicand, can economically be done on the abacus, by making out a table of the products by the nine digits, and then adding on the instrument the products for each digit of the multiplier, moving one place to the right each time; and if only a certain number of significant figures are wanted, nothing need be added to the right of a certain column. Similar use may be made of the abacus in connection with the various printed partial product tables of interest, etc.

The use of the suan p'an is taught in elementary schools in China by means of a large demonstration instrument, analogous to our demonstration slide rules, which is set on the wall, the beads being prevented from sliding down by bristles set in the rods. Every shopkeeper, and even illiterate coolies, can use it to a certain extent, with great variation in accuracy and speed among individuals, just as there is in written or mental computation with us. Chinese bank clerks are most expert in it, and can go through long calculations of money exchange, which involve two long divisions, such as £ 3,567 5s. 9d. at 6s. 3½d. = Shanghai Taels ? at .731 = Mexican Dollars ?, and give the answer in about the time that it takes us to set down the figures.

While the westerner can probably not hope for such speed, and indeed if he had occasion for it, it would probably be more economical for him to buy an adding machine rather than to spend his time developing it, still the simplicity and cheapness of the instrument might make it a useful help to anyone who has a moderate amount of addition to do, whether in accounting, or in any mathematical work.

CLUB ACTIVITIES.

The Mathematical Club of Smith College, Northampton, Mass.

[*1918*, 91, 455].

Officers of the club for the year 1918–19 were: President, Professor Eleanor P. Cushing; vice-president, Barbara Johnson '19; secretary, Emily Porter '19; treasurer, Cornelia Hopkins '19.

November 25, 1918: "Work in the New York Life Insurance Company" by Professor Ruth Wood.

December 2: "Theories of determinants" by Eleanor Smith '19.

December 18: Christmas party.

February 10, 1919: "Theory of determinants" (continued) by Marjorie Stanton '19.

March 3: "Permutations" by Adele Adams '19; "Combinations" by Helen Crittenden '19.

March 24: "Theory of Probability" by Frances Lowe '19 and Gladys Kern '19.

April 28: "Theory of Probability" (continued) by Florence Fessenden '19.

May 19: Social meeting.

The officers elected for the year 1919–20 are: President, Professor Cushing; vice-president, Mary F. McConnaughy '20; secretary, Constance Torrey; treasurer, Ruth Andrew '20.

PI MU EPSILON FRATERNITY, Syracuse University, Syracuse, N. Y. [*1918*, 271–273].

During the year 1918–19 there were forty-one members, of whom sixteen were faculty and graduates and twenty-five were undergraduates. The officers were: Director, Professor John L. Jones; vice-director, Professor Louis Lindsey; secretary, Gertrude Reynolds '19; treasurer, Donald F. Sears '20; librarian, Agnes Wilcox '20; executive committee, the above officers and Roy Horst '19, Helen De Long '19, Ora M. Tanner '19; scholarship committee, Professors Floyd F. Decker and William H. Metzler, Roy Horst '19, Ethel M. Hicks '19 and Lona Preston '19.

December 2, 1918: Outline of plan for the year. Discussion of mathematical magazines.

January 6, 1919: Report of the scholarship committee and election of new members.

January 27: Initiation of new members. "Rerating of regent's papers" by Professor Lindsey.

February 17: "The method of least squares" by Joseph Atwell '19; "An application of the binomial theorem" by William Start '19.

March 10: "Normals to conics" by Ora Tanner '19, Cornelia Tyler '19 and Bertha Adams '19.

March 31: "The planimeter and how to integrate by mechanical means" by Professor Street.

April 28: "Teaching graphs in high school" by Professor Lindsey.

May 12: Election of officers. Informal talks by the faculty and seniors.

May 14: Annual picnic.

May 26: Special meeting to vote on the establishment of a chapter at Ohio State University. (Note: A new chapter of Pi Mu Epsilon has been established at Ohio State University.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2822. Proposed by A. M. HARDING, University of Arkansas.

Show that the sum of the series

$$1 + 3\cdot 2 + 5\cdot 2^2 + 7\cdot 2^3 + \cdots + (2n-1)2^{n-1}$$

to n terms is $3 - 2^n + (n-1)2^{n+1}$.

2823. Proposed by S. A. COREY, Des Moines, Iowa.

Let TQ and PR be diameters of a circle with center O. Bisect TO at X and draw PQ. On PQ erect the perpendicular XW and on PR, the perpendicular QV. Prove that $OX \cdot PV = PW \cdot PQ$.

2824. Proposed by G. Y. SOSNOW, Newark, N. J.

If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3, the lengths of the three tangents drawn from any point to the semi-cubical parabola, $ay^2 = x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$. From *Mathematical Tripos Examination*, Cambridge, England.]

2825. Proposed by the late L. G. WELD.

A ball, having a coefficient of resilience α, strikes a rigid plane surface, inclined at an angle θ from the horizontal, after falling through a height h. What is the distance from the first to the second point of impact with the plane?

2826. Proposed by ALBERT A. BENNETT, University of Texas.

As a standard form for a square non-singular symmetric matrix under certain transformations, may be taken the form in which only the elements in the secondary diagonal are different from zero, and each of these is equal to unity. Analogously, as a standard form for a square non-singular skew-symmetric matrix (and hence incidentally of even order), may be taken the form in which only the elements of the secondary diagonal are different from zero, while the half of these which are towards the upper right-hand corner are each minus one, and the remaining half towards the lower left-hand corner, are each plus one. Denote both of these standard matrices by N.

Give simple parallel proofs that if M be given as non-singular and symmetric or non-symmetric as the case may be, a matrix P exists such that, with the usual notation

$$M = PNP'.$$

2827. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle having a given base and a given altitude.

2828. Proposed by T. M. BLAKSLEE, Ames, Iowa.

On page 72 of R. B. Hayward's *The Algebra of Coplanar Vectors and Trigonometry* occurs the sentence: "It will be a good exercise for the student to show that $\cos(90°/7) = \frac{1}{2}\sqrt{x_1}$, where x_1 is the greatest root of the equation,

$$x^3 - 7x^2 + 14x - 7 = 0."$$

(1) Do not merely verify but deduce the equation and find x_1. (2) Deduce the x-equation (x_1, x_2, x_3, x_4, the roots) such that its greatest root x_1 gives $\cos(90°/9) = \cos 10° = \frac{1}{2}\sqrt{x_1}$. (3) Of what angles are $\frac{1}{2}\sqrt{x_1}, \cdots \frac{1}{2}\sqrt{x_4}$, in (2), the cosines? Develop a method of writing out at once $\cos(nv)$ in terms of powers of $\cos v$ if these are given for $(n-1)v$ and $(n-2)v$. The same for $\sin(nv)$. (4) Use the results of (2) and (3) to find the number of degrees in a radian. Hence, find π from radian instead of radian from π as is usual.

SOLUTIONS OF PROBLEMS.

411 (Algebra) [1914, 121; 1919, 268, 459]. Proposed by V. M. SPUNAR, Chicago, Ill.

Determine $x_1, x_2, x_3 \cdots x_p$ from the equations:

$$\begin{aligned}
x_1 + x_2 \quad &+ x_3 \quad + \cdots + x_p \quad &= a_0,\\
b_1x_1 + b_2x_2 \quad &+ b_3x_3 \quad + \cdots + b_px_p \quad &= a_1,\\
b_1^2x_1 + b_2^2x_2 \quad &+ b_3^2x_3 \quad + \cdots + b_p^2x_p \quad &= a_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
b_1^{p-1}x_1 + b_2^{p-1}x_2 \quad &+ b_3^{p-1}x_3 \quad + \cdots + b_p^{p-1}x_p \quad &= a_{p-1}.
\end{aligned}$$

A solution has been sent in by P. J. DA CUNHA, University of Lisbon, Portugal, in which the analysis is very much the same as that previously printed, but he also considers the case when some of the b's are equal. For example,

if $p = 3$ and $b_1 = b_2 \gtrless b_3$, we must have

$$\begin{vmatrix} 1 & 1 & a_0 \\ b_1 & b_3 & a_1 \\ b_1^2 & b_3^2 & a_2 \end{vmatrix} = 0.$$

If this condition is satisfied, the expressions for x_1 and x_2 in the general solution take indeterminate forms, and there will be an infinite number of solutions given by

$$x_1 + x_2 = \frac{a_0b_3 - a_1}{b_3 - b_1}, \qquad x_3 = \frac{a_0b_1 - a_1}{b_1 - b_3}.$$

463A (Geometry) [1915, 161; 1919, 414]. Proposed by B. J. BROWN, Kansas City.

If μ and ν are the parameters of the two confocal conicoids through any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

show that $\mu+\nu+a^2+c^2=0$, along a central circular section.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The parameters being μ and ν, the conicoids confocal with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

are

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}+\frac{z^2}{c^2+\mu}=1 \tag{2}$$

and

$$\frac{x^2}{a^2+\nu}+\frac{y^2}{b^2+\nu}+\frac{z^2}{c^2+\nu}=1. \tag{3}$$

(1) − (2) gives

$$\frac{x^2}{a^2(a^2+\mu)}+\frac{y^2}{b^2(b^2+\mu)}+\frac{z^2}{c^2(c^2+\mu)}=0 \tag{4}$$

and (1) − (3),

$$\frac{x^2}{a^2(a^2+\nu)}+\frac{y^2}{b^2(b^2+\nu)}+\frac{z^2}{c^2(c^2+\nu)}=0. \tag{5}$$

The equation of the central circular sections of (1) is

$$c^2(a^2-b^2)x^2-a^2(b^2-c^2)z^2=0, \tag{6}$$

$a^2>b^2>c^2>0$. Since (4), (5), and (6) must be consistent for values of x, y, z, not all zero,

$$\begin{vmatrix} \frac{1}{a^2(a^2+\mu)}, & \frac{1}{b^2(b^2+\mu)}, & \frac{1}{c^2(c^2+\mu)} \\ \frac{1}{a^2(a^2+\nu)}, & \frac{1}{b^2(b^2+\nu)}, & \frac{1}{c^2(c^2+\nu)} \\ c^2(a^2-b^2), & 0, & -a^2(b^2-c^2) \end{vmatrix}=0.$$

After removing the factor $\mu-\nu\neq 0$, and certain other factors which are not zero by the conditions above, this determinant reduces to

$$a^2+c^2+\mu+\nu=0.$$

499 (Geometry) [1916, 341; 1919, 414]. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

SOLUTION BY W. D. CAIRNS, Oberlin College.

With any position of three mutually perpendicular axes, for a fixed value of x the equations of the surfaces must be of the form $l_1(y^2+z^2, y, z)=0$, where l_1 designates a linear form, the coefficient of (y^2+z^2) not vanishing; similarly, for $y=$ const. and for $z=$ const. they must take the forms $l_2(x^2+z^2, x, z)=0$ and $l_3(x^2+y^2, x, y)=0$, respectively. It may be shown by theorems relating to the equivalence of polynomials that this is possible only if the equations of the surfaces are of the linear form:

$$l(x^2+y^2+z^2, x, y, z)=0,$$

in which the coefficients are constants; that this is sufficient is a well known fact. Hence, the class of surfaces is composed of all spheres.

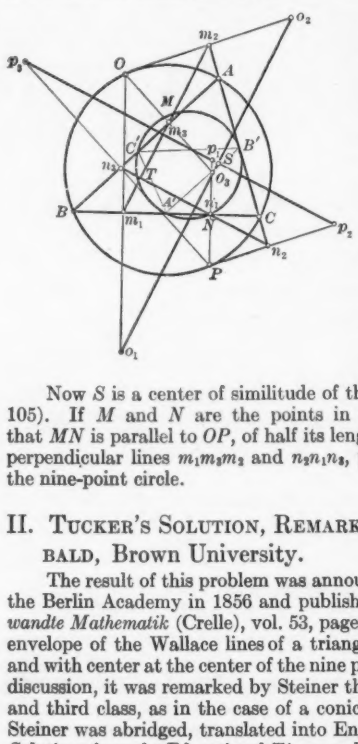
Also discussed by WILLIAM HOOVER.

2776 [1919, 267]. Proposed by C. P. SOUSLEY, State College, Penn.

Prove by elementary geometry that the Wallace lines of the extremities of any diameter of the circumscribed circle of a triangle intersect at right angles on the nine-point circle of the triangle.

I. Solution by F. V. Morley, Johns Hopkins University.

Let ABC be the given triangle, O and P the ends of any diameter of the circumcircle. Let p_1, p_2, p_3, be the images of P in the sides of the triangle; then $p_1p_2p_3$ is the *line of images* of P, and $n_1n_2n_3$ the corresponding Wallace line. And similarly for O.



Any line of images passes through S, the orthocenter. For PS is known to be bisected by the Wallace line $n_3n_1n_2$[1] (Casey, *Sequel to Euclid*, second edition, 1882, p. 36). Hence, S is on $p_3p_1p_2$. Similarly, S is on any line of images (l) of a point (p) on the circumcircle.

Consider p as moving round the circle. Then l pivots on S. The angle BSC is the supplement of the angle A, and therefore the image of the circumcircle in BC will pass through S. As p moves through an arc $2x$ along the circumcircle, its image p_1 will move through an equal arc along the circle BSC. The line p_1S, namely l, will rotate about S through the angle x. The rotation of l is in the contrary sense to the motion of p, and of half the measure. Hence, when p moves through an angle π, as from P to O, l moves through the angle $\pi/2$. Therefore $p_1p_2p_3 \perp o_1o_2o_3$.

It follows that the Wallace lines for P and O are perpendicular, for they are parallel to the lines of images.

Now S is a center of similitude of the circumcircle and the nine-point circle (Casey, *l.c.*, p. 105). If M and N are the points in which OS and PS cut the nine-point circle, it follows that MN is parallel to OP, of half its length, and hence a diameter of the nine-point circle. The perpendicular lines $m_1m_2m_3$ and $n_3n_1n_2$, passing through M and N, will therefore meet in T on the nine-point circle.

II. Tucker's Solution, Remarks, and Historical Notes. By R. C. Archibald, Brown University.

The result of this problem was announced, without proof, by Steiner in a memoir read before the Berlin Academy in 1856 and published the following year in *Journal für die reine und angewandte Mathematik* (Crelle), vol. 53, pages 231–237. It was here also that Steiner stated that the envelope of the Wallace lines of a triangle was a three cusped hypocycloid tangent to its sides, and with center at the center of the nine point circle. Hence, from the result of our problem under discussion, it was remarked by Steiner that the orthoptic locus of this curve of the fourth degree and third class, as in the case of a conic section, is a circle (in part, at least). This memoir of Steiner was abridged, translated into English, and published in *Mathematical Questions with their Solutions from the Educational Times*, vol. 3, 1865, pp. 97–100.

Many solutions of the problem proposed by Dr. Sousley have been published. Three, including a brief one by R. Tucker, were published in the volume of *Mathematical Questions* mentioned above, pages 58–59. Others are: by Le Bel and Talayrach, *Nouvelles annales de mathématique*, April, 1865, pp. 177–178; by E. Lemoine in *Journal de mathématiques élémentaires* (Bourget), 1883, pp. 246–247; by Goffart in *Nouvelles annales de mathématique*, August, 1884, pp. 397–399; and by Weill in *Journal de mathématiques spéciales*, 1884, pp. 15–16.

Tucker's solutions is as follows:

"Since Om_1B, Om_2B, Pn_2B, Pn_1B are all right angles Om_1m_2, Pn_2n_1 are equal respectively to OBm_2, PBn_1; and OBP is a right angle; therefore m_2Tn_1 is a right angle.

Again let A', C' be the middle points of BC, BA and therefore also of m_1n_1, n_2m_2; then the angles $A'Tn_1$, $C'Tn_2$ are equal respectively to $A'n_1T$, $C'n_2T$, consequently the angle $A'TC'$ is

[1] This result seems to have been first formulated by Retsin, *Nouvelles annales de mathématiques*, vol. 28, 1869, p. 530—Editor.

supplemental to the angle ABC or $C'B'A'$, B' being the middle point of AC; hence T lies on the circle through A', B', C', that is to say, on the nine-point circle of the triangle ABC."

In *The Mathematician*, vol. 2, pp. 289–290, July 1847, Fenwick discussed the nine-point conic[1] which may be defined as follows: Given a triangle ABC and a point P in its plane, a conic can be drawn through the following nine points: (1) The middle points of the sides of the triangle; (2) the middle points of the lines joining P to the vertices of the triangle; (3) the points where these last named lines cut the sides of the triangle. Fenwick proved that this conic is the locus of the center of the conics passing through the four points A, B, C, and P; when P is the orthocenter of the triangle the conic is a rectangular hyperbola. Brianchon and Poncelet proved (*Annales de mathématiques pures et appliquées* (Gergonne), tome 11, pp. 205, 213, January, 1821) that every rectangular hyperbola through A, B, and C must also pass through the orthocenter and that the locus of the centers of such hyperbolas is the nine-point circle of the triangle. The asymptotes for any one of the hyperbolas are the pair of Wallace lines corresponding to the ends of a certain diameter of the circumscribed circle. (Steiner, *l.c.*, cf. also Weill, *Journal de mathématiques spéciales*, 1884, p. 16.) When this diameter passes through the center of the inscribed circle we have that the corresponding Wallace lines intersect at the point of contact of the nine-point and inscribed circles (Neuberg, *Mathesis*, 1893, p. 86). The equilateral hyperbola with this point as center has been called the *hyperbola of Feuerbach* (Neuberg, *l.c.*) since the theorem that the nine-point circle of a triangle is tangent to the inscribed and the three escribed circle was enunciated and proved by Feuerbach in a publication of 1822. I have not seen a reference to the three similar hyperbolas corresponding to diameters through the centers of the escribed circles.

Corresponding to the diameter through the orthocenter (that is, Euler's line) we have the *hyperbola of Jeřábek* (Jeřábek, *Mathesis*, 1888, pp. 81–84), and to the diameter through the Lemoine point,[2] we have the hyperbola called by Neuberg (*Journal de mathématiques spéciales*, 1886, p. 73) *hyperbola of Kiepert* (*Nouvelles annales de mathématique*, 1869, pp. 40–42.).

A more general form of Dr. Sousley's problem was given by Weill (*Journal de mathématiques spéciales*, 1884, p. 13): If m and m' are the Wallace lines of any two points M and M' on the circumcircle, then the angles between m and m' are equal to one or other of the angles under which the line segment MM' is seen from a point of the circumcircle.

Also solved by P. J. DA CUNHA, A. PELLETIER, and the Proposer.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. W. H. WILSON, of the Massachusetts Institute of Technology, has been instructor in mathematics at the University of Iowa since last September.

At the University of Oklahoma Associate Professor E. P. R. DUVAL resigned, his resignation to become effective at the end of the first semester, and is engaged in fruit growing at Springdale, Arkansas. Mr. E. E. COWAN has been appointed instructor in mathematics.

At Yale University Assistant Professor W. R. LONGLEY has been promoted to a full professorship, and Instructor J. K. WHITTEMORE to an assistant professorship of mathematics.

[1] This name seems to be due to Beltrami, 1863, "Intorno alle coniche dei nove punti e ad alcune questioni che ne dipendone," *Memorie dell'accademia delle scienze dell' Istituto di Bologna*, serie 2, vol. 2 (1862), pp. 361–395; also *Opere matematiche di Eugenio Beltrami*, vol. 1, 1902, pp. 45–72.

[2] This is the point of intersection of the symmedians of a triangle, that is, the lines through the vertices of the triangle symmetric to the medians through those vertices, with respect to the corresponding angle bisectors.

Assistant Professor A. E. BABBITT, of the University of Nebraska, has resigned his position in the mathematics department, and has accepted the position as secretary and actuary with the Lamar Insurance Company of Jackson, Miss.

At the University of Texas are the following part-time instructors in mathematics this year in the school of pure mathematics: Messrs. J. E. BURNAM, H. H. HAMMER, CLAUDE BAILEY, and C. H. ROBERSON; Mr. C. M. CLEVELAND was appointed instructor in mathematics in the school of applied mathematics.

At the University of Kansas, Assistant Professor SOLOMON LEFSCHETZ has been advanced to an associate professorship.

Mr. P. C. PORTER is dean of the college and professor of mathematics in Baylor College for Women, Belton, Texas.

P.ERRE ROBINSON, now a graduate student at the University of Chicago, has accepted an instructorship in mathematics at the University of Chicago High School.

Dr. I. A. BARNETT, now Benjamin Peirce instructor at Harvard University, will go to University of Illinois as instructor in mathematics next year.

Mr. C. C. WYLIE who, aside from war work with the Technical Staff of the Ordnance Department, has for six years been an assistant in the U. S. Naval Observatory, has taken a position in the department of astronomy in the University of Illinois.

The official title of Professor E. R. HEDRICK, of the University of Missouri, has been changed from professor of mathematics to professor of mathematics and of the teaching of mathematics.

At Washington University, Miss JESSICA M. YOUNG, instructor in Astronomy at Northwestern University during the first semester, has been instructor in mathematics and astronomy since the beginning of the second semester; Mrs. PEARL C. MILLER has been an assistant in the mathematics department since the middle of last December.

Professor E. W. BROWN, of Yale University, whose term of office as a representative of the American Mathematical Society on the Division of Physical Sciences in the National Research Council would have expired this month, resigned in December. The Council of the Society appointed as his successor Professor OSWALD VEBLEN of Princeton University.

The government of Servia has created a university of Lioubliana (formerly Laibach) and Professors J. PLEMELJ, of the University of Czernowitz (Roumania), and R. ZOUPANTCHITCH (French form of Suppantschitch), honorary docent of the polytechnic school at Vienna, have been appointed professors of mathematics.

At the University of Clermont Dr. G. GIRAUD has been appointed chargé de cours for differential and integral calculus in place of Professor A. C. E. PELLET who has retired from active teaching. At the University of Poitiers Dr. RENÉ GARNIER has been appointed chargé de cours for rational and applied mechanics in place of Professor M. FRÉCHET now at the University of Strassburg (*1919*, 371).

The following paragraph is from *Science*, February 13: "Boston University has concluded an arrangement for an exchange of professorships in mathematics for the college year 1920–1921 with Tsing Hua College, Peking, China. Professor R. E. Bruce, chairman of the department in Boston University, will exchange with Professor A. H. HEINZ, of Tsing Hua. Professor HEINZ, head of the department of mathematics, is a graduate of the University of Missouri and has been at Tsing Hua nine years. This college is under the control of the Chinese government and was founded with part of the returned Boxer Indemnity. Professor Bruce will sail from the Pacific coast in April. Professor Heinz will reach this country in time to begin his work at Boston University at the opening of the college in September."

Dr. ADOLF HURWITZ, professor of mathematics in the federal polytechnic school at Zurich, since 1892, died in November 1919, aged sixty years.

Nature for December 25, 1919, notes:

"The last day of this year marks the bicentenary of the death of JOHN FLAMSTEED, first Astronomer Royal of England, and the rector of the parish of Burstow, Surrey, where he is buried, uncommemorated, we understand, by an monument. Flamsteed was born four years after Newton, and was a native of Derbyshire, being the son of a well-to-do maltster. Though prevented by illness from attending a university, he was devoted to mathematical studies, and in 1671 sent a paper to the Royal Society. Three years later he published his 'Ephemerides,' a copy of which, being presented to Charles II. by Sir Jonas Moore, led to Flamsteed being appointed on March 4, 1675, 'our Astronomical Observer' at a salary of 100£ per annum, his duty being 'forthwith to apply himself with the most exact care and diligence to the rectifying the tables of the motions of the heavens and the places of the fixed stars, so as to find out the so much desired longitude of places for the perfecting the art of navigation.' The observatory at Greenwich, constructed partly of brick from old Tilbury Fort and of timber and lead from the Tower of London, was designed by Wren and built at a cost of 520£, the money being derived from the sale of spoilt gun-powder. The struggles and disputes, the dogged perseverance, and the memorable achievements of Flamsteed have their place in the history of astronomy, but it may safely be said that never has king or Government made a better investment than when Greenwich was built and Flamsteed made passing rich on 100£ a year."

We have already noted (*1920*, 45) the formation of a committee, or "Provisional International Mathematical Union" charged with the duty of consulting mathematicians and circulating statutes in different countries with a view to organizing an International Mathematical Union. C. J. de la VALLÉE POUSSIN was appointed the president of this committee, W. H. YOUNG the vice-president, and LAMB, PICARD, and VOLTERRA honorary presidents; there were four "secretaries" representing Belgium, France, Roumania and Italy. The Italian secretary, Professor Reina, died last November (*1920*, 143). In December, 1919, five months after the Brussels Congress, American mathematicians learned for the

first time of the appointment of such a provisional committee. On December 30 the Council of the American Mathematical Society appointed Professor L. E. DICKSON chairman of a committee to deal with communications coming from the provisional union. It has developed, however, that at an earlier date the union (with its secretaries representing Belgium, France and Roumania) called for an International Mathematical Congress (so-called), at the University of Strasbourg, the same to commence September 22, 1920. According to information recently received it would appear as if English mathematicians were with Americans equally ignored in the decision with regard to the desirability of a Congress at the time stated.

The following reports of Summer Sessions in 1920 have been received.

University of Chicago, First Term, June 1–July 28; Second Term, July 29–September 3. By Professor E. H. MOORE: Hermitian matrices of positive type in general analysis, 4 hours, Limits and Series, 4 hours, first term only. By Professor G. A. BLISS: Functions of a complex variable, 4 hours; Integral calculus, 5 hours. By Professor L. E. DICKSON: Theory of matrices and bilinear and quadratic forms, 4 hours; Theory of numbers, 4 hours. By Professor M. W. HASKELL: Projective geometry, 4 hours; Topics in analytic geometry, 5 hours. By Professor J. W. A. YOUNG: Theory of equations, 4 hours; Differential calculus, 5 hours. By Professor E. W. CHITTENDEN: Differential equations, 4 hours; College Algebra, 5 hours. By Professor W. D. MACMILLAN: Celestial mechanics II, 4 hours; Descriptive astronomy, 5 hours. Courses in plane analytic geometry and trigonometry will also be given.

University of Colorado, First term, June 14–July 21; second term, July 22–August 28. By Professor ABRAHAM COHEN, Johns Hopkins University: Teachers course in mathematics; Differential equations. By Professor G. H. LIGHT: Review course in mathematics; Calculus of variations. By Dr. GUY W. SMITH, University of Kentucky: Trigonometry; Plane analytical geometry; Differential calculus. The following advanced courses will be given by Professors COHEN and LIGHT: Theory of algebraic equations; Definite integrals; Theory of a complex variable; Elliptic integrals and functions; An introductory course in analysis; Differential geometry.

Columbia University, July 6–August 13. Undergraduate courses in elementary algebra, plane geometry, logarithms and trigonometry, solid geometry, college algebra, analytical geometry and calculus will be given by various members of the staff. Graduate courses are offered as follows: By Professor E. KASNER: Fundamental concepts of modern mathematics; Geometric transformations and groups. By Professor W. B. FITE: Differential equations. By Dr. G. E. PFEIFFER: Introduction to higher algebra. In the School of Education, Teachers College, several courses will be given by Professor UPTON and Mr. BRECKENRIDGE for teachers of mathematics in secondary schools. Professor SMITH will lecture on the history of mathematics and give a practicum in the teaching of mathematics.

Harvard University, July 6–August 14. By Professor C. L. BOUTON: Trigonometry; Analytic geometry. By Professor G. D. BIRKHOFF: Differential calculus; Integral calculus. These courses will be accepted as half-courses towards the degrees of A.B., A.A., and S.B.

University of Illinois, June 21–August 14, By Professor A. R. CRATHORNE: Mathematical theory of statistics; Differential equations. By Professor A. J. KEMPNER: Functions of a complex variable; Advanced algebra. By Dr. E. B. LYTLE: Teachers' course; Plane trigonometry. By Dr. J. R. KLINE: Differential calculus. By Mr. W. E. EDINGTON: Integral calculus; College algebra. By Mr. PETTIT: Trigonometry; College algebra.

University of Iowa, First Term, June 16–July 27. By Professor H. L. RIETZ: Teachers' course; Solid geometry. By Dr. J. W. CAMPBELL: Trigonometry; Analytic geometry; Differential equations. By Dr. W. H. WILSON: College algebra; Calculus; Reading course. By Mr. R. E. GLEASON: Solid geometry. Second Term, July 28–August 31. By Professor J. F. REILLY: Analytic geometry; Calculus; Reading course. By Mr. R. E. GLEASON: College algebra; Reading course. The courses are five hours per week in class through the sessions. When transformed into hours of the ordinary academic year the credit for each course taken during the first session is two semester hours and for the second session five-sixths of this amount.

University of Kansas, June 10–July 31. By Professor C. H. ASHTON: Mechanics, 3 semester hrs. credit; Algebra, 3 hrs. By Professor S. LEFSCHETZ: Theory of numbers, 3 hrs.; Analytic geometry, 2 hrs. By Professor J. J. WHEELER: Calculus, 3 hrs.; Solid geometry, 3 hrs. Second session, July 22–August 19. By Professor E. B. STOUFFER: Mathematical theory of investments, 2 hrs.; Trigonometry, 2 hrs.

University of Michigan, June 28–August 20. By Professor BEMAN: Differential equations, 2 hours. By Professor W. B. FORD: Advanced algebra; Advanced calculus, 2 hours; Theory of potential, 2 hours. By Professor P. FIELD: Vector analysis, 2 hours. By Professor L. C. KARPINSKI: History of mathematics. By Professor J. W. Bradshaw: Projective geometry, 2 hours. By Professor T. R. RUNNING: Graphical methods, 2 hours. By Professor W. B. CARVER: Introduction to the mathematical theory of interest, 2 hours. By Dr. R. B. ROBBINS: Mathematical theory of statistics, 2 hours.

Northwestern University, June 28–August 20. By Professor D. R. CURTISS: College algebra, Differential calculus. By Professor C. E. WILDER: Trigonometry; Analytical geometry. Each course bears credit of 3 semester hours.

University of Oklahoma, June 9–August 3. By Professor REAVES: Analytic geometry, 5 hrs.; Advanced analytic geometry, 3 hrs. By Professor GOSSARD: Intermediate algebra, 3 hrs.; The teaching of secondary mathematics, 2 hrs.; Integral Calculus, 3 hrs. By Professor N. ALTSHILLER-COURT: College, algebra, 3 hrs.; Plane trigonometry, 3 hrs. Projective geometry, 3 hrs. By Professor E. D. MEACHAM: Differential calculus, 3 hrs.; Solid analytic geometry, 3 hrs.

University of Pennsylvania, July 5–August 14. Courses in Plane geometry;

Differential calculus; Integral calculus; Solid geometry; Plane trigonometry; Intermediate algebra; College algebra; Analytic geometry; Projective geometry; Theory of statistics; will be offered by the following staff: Professor G. H. HALLETT, Assistant Professor H. H. MITCHELL, Assistant Professor S. P. SHUGERT and Mr. THOMAS.

Queen's University, July 5–August 13. Freshman and Sophomore Mathematics will be given by the following staff: Professor J. MATHESON; Professor D. BUCHANAN; Professor C. F. GUMMER; Messrs. N. MILLER, H. VAN PATTER and F. M. WOOD; and Miss GRACE H. JEFFREY.

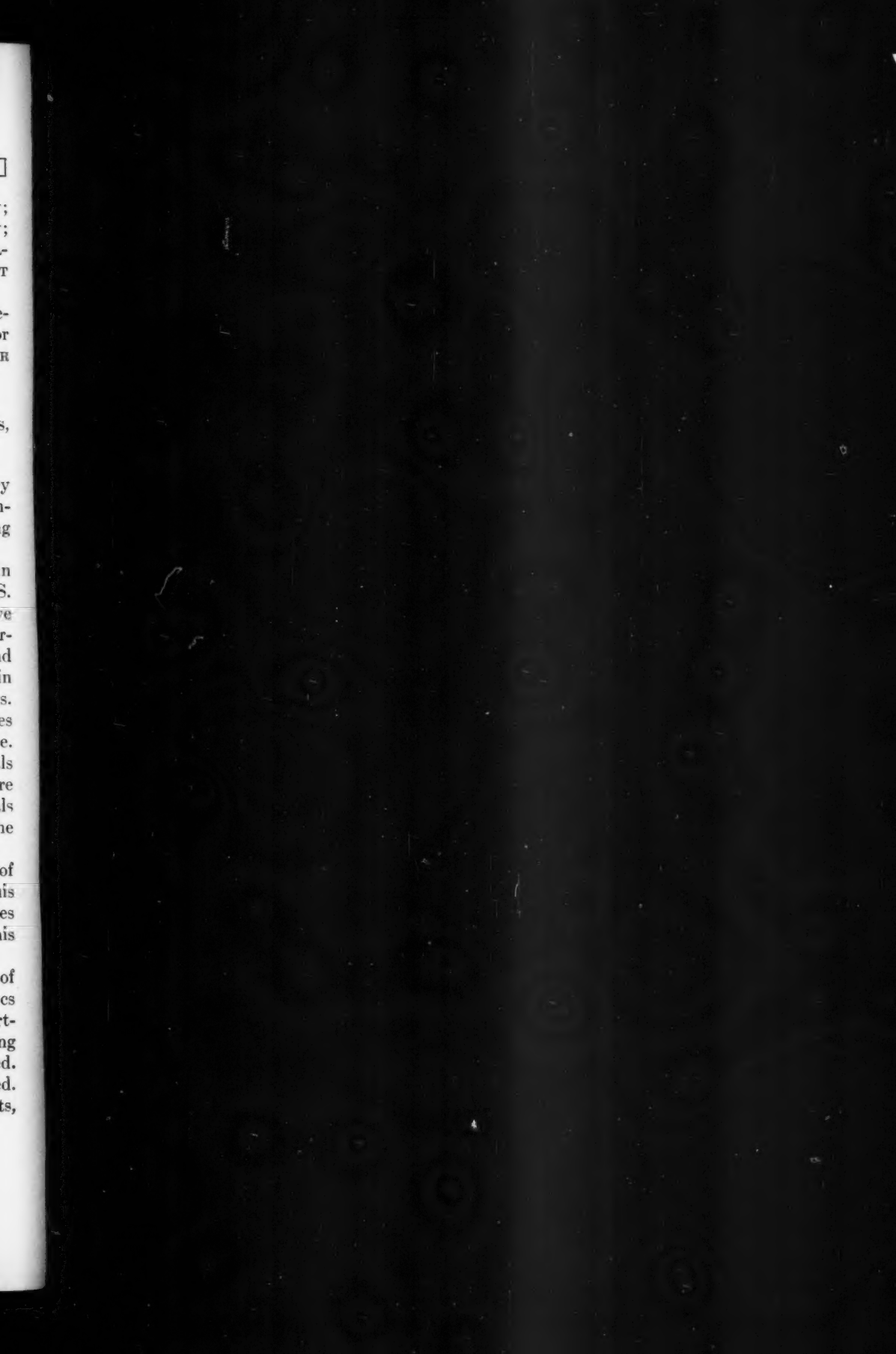
THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS, MARCH 1920. (Cf. *1920*, 145–146.)

At the last meeting of the General Education Board in New York on February 28th, the sum of $25,000, was appropriated for the use of the National Committee on Mathematical Requirements to continue its work for the year beginning July 1, 1920.

The preliminary report on "The Reorganization of the First Courses in Secondary School Mathematics" published for the Committee by the U. S. Bureau of Education has been distributed widely. Copies of the report have gone to all the state departments of education, to all county and district superintendents in the United States and to all city superintendents in cities and towns of over 2500 population. It has been sent to all the normal schools in the country, to some 1500 libraries and to almost 300 periodicals and newspapers. In addition it has been sent to about 4500 individuals, the names and addresses of which were furnished the Bureau of Education by the National Committee. This list of individuals consists chiefly of teachers of mathematics and principals of schools throughout the country. Additions to this mailing list to secure copies of the future reports of the Committee can still be made. Individuals interested in securing these reports should send their names and addresses to the Chairman of the Committee (J. W. Young, Hanover, N. H.).

A subcommittee consisting of Professor C. N. Moore of the University of Cincinnati, Mr. W. F. Downey of Boston and Miss Eula Weeks of St. Louis has been appointed to prepare a report for the Committee on Elective Courses in Mathematics for Secondary Schools. Any material or suggestions for this report may be sent directly to any member of the subcommittee.

The recent work of the National Committee had a place on the program of the organization meeting of the National Council of Teachers of Mathematics held in Cleveland on February 24, in connection with the meeting of the Department of Superintendence of the National Education Association. The meeting for the organization of the National Council was enthusiastically attended. A constitution was adopted and officers and an executive committee elected. Mr. J. A. Foberg, of the National Committee on Mathematical Requirements, was elected Secretary-Treasurer of the National Council.



A MECHANISM FOR THE SOLUTION OF AN EQUATION OF THE *N*TH DEGREE.

By A. L. CANDY, University of Nebraska.

W. Peddie[1] read a paper on this subject before the Fifth International Congress of Mathematicians held in Cambridge, England, in 1912. This paper is found in the *Proceedings* of this Congress, Vol. I, pp. 399–402. Professor Peddie's communication also contains a picture of his mechanism. From this picture, as well as from his description, it is quite evident that this instrument can not be manufactured by any one except an expert mechanic, who has access to a shop well equipped for working in metal. The crude instrument, that I shall describe and explain herein, was made by myself. A similar one can be made by any one who has a moderate amount of mechanical ingenuity and a few simple tools, out of materials that can be procured almost anywhere. Both the design and the proof that I here submit, seem to me to be simpler, although the underlying principle is virtually the same as that used by Professor Peddie.

Description of the Mechanism. This mechanism, shown closed in Fig. 1, and open in Fig. 2, consists of a main bar about 32 inches long, to which are hinged three arms each about 8 inches long, the distances between the hinges being equal. A lighter connecting bar is attached to the free ends of the arms in such a manner that these arms always turn through the same angle. On the main bar, and also each of the arms, are beveled cleats along which grooved slides move freely. Each of these slides on the main bar carries an eye-headed screw, like those used to fasten the hanging cords to picture frames. These screws are placed so that when the instrument is closed and these slides are at their zero points, as shown in Fig. 1, the eyes are in line with the pins of the hinges. Each slide on the arms carries a small drum that is held firm by means of a milled nut. To each of the drums is attached a small, flexible, inelastic cord, which passes through the eye carried by the adjacent slide on the main bar, and is fastened to the next slide below on the main bar, the lower end of the last cord being made fast to the main bar. The first slide is held in place by means of a small iron pin inserted in holes in the main bar. A graduated circular scale is placed under the first arm, from which the roots of the equation are read. The scales for reading the positions of the slides are marked off on the left side of the main bar. The instrument may be used (1) in a vertical position, as shown in the figures, so that the lengthening of any string by unwinding will cause some of the slides to move downwards by their own weight; or (2) lying on a table and operated with both hands.

Solution of an Equation. Let us now solve the equation

$$10x^3 + 24x^2 + 9x - 7 = 0. \qquad (1)$$

The process is as follows: First, close the instrument, then wind up the drums

[1] Professor of physics, University College, Dundee.